

# *Quasidiagonal Operators, Approximation, and $C^*$ -Algebras*

K. R. DAVIDSON, D. A. HERRERO & N. SALINAS

**Introduction.** An operator  $T$ , acting on a complex, separable, infinite dimensional Hilbert space  $\mathcal{H}$ , is called *quasidiagonal* [12] if there is an increasing sequence  $\{P_n\}_{n=1}^\infty$  of finite rank orthogonal projections such that

$$P_n \rightarrow 1 \text{ strongly, and } \|TP_n - P_nT\| \rightarrow 0 \quad (n \rightarrow \infty).$$

$T$  is called *quasitriangular* if there is a sequence  $\{P_n\}_{n=1}^\infty$  as above such that

$$\|(1 - P_n)TP_n\| \rightarrow 0 \quad (n \rightarrow \infty);$$

$T$  is *biquasitriangular* if both  $T$  and its adjoint  $T^*$  are quasitriangular. It is easily seen that a quasidiagonal operator is biquasitriangular. Moreover, biquasitriangular operators are the norm-limits of the algebraic operators [30] (see also [14, Chapter 3]), and therefore every quasidiagonal operator is the norm-limit of algebraic ones.

**Problem 1.1.** *Is every quasidiagonal operator the norm-limit of algebraic quasidiagonal operators?*

The partial answer to this and closely related problems are a good indication of what we really know about the meaning of quasidiagonality. Our main result in the positive direction says that if  $T$  is quasidiagonal, the  $C^*$ -algebra generated by  $\tilde{1} = \pi(1)$  and  $\tilde{T} = \pi(T)$  admits a faithful unital  $*$ -representation  $\rho$  on a separable space  $\mathcal{H}_\rho$  such that  $\rho(\tilde{T})$  is quasidiagonal, and the essential spectrum of  $T$  does not disconnect the plane; then  $T$  is, indeed, the norm-limit of algebraic quasidiagonal operators. (Here  $\pi : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$  denotes the canonical projection from the algebra  $\mathcal{L}(\mathcal{H})$  of all (bounded linear) operators acting on  $\mathcal{H}$ , onto the quotient Calkin algebra  $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ , where  $\mathcal{K}(\mathcal{H})$  denotes the ideal of all compact operators.)

Several other results will be interpolated in the following survey.