

Probability and the Equivalence of Generalized H^p Spaces

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1. Introduction. In this paper we show that several generalizations of the H^p spaces are identical, and thereby answer a question of P. A. Meyer [4]. Thanks to our probabilistic methods, we are able to study spaces of functions which are harmonic with respect to degenerate operators.

Traditional H^p theory studies harmonic functions u , for which $\Delta u = 0$. There are two ways of reconstructing such a function defined on

$$\mathbf{R}_+^{n+1} = \{(t, x_1, \dots, x_n) \in \mathbf{R}^{n+1} : t \geq 0\}$$

from its values on the boundary corresponding to $t = 0$. The first method uses the Poisson kernel, and the second uses Brownian motion:

$$u(t, x_1, \dots, x_n) = E\{u(B_\tau) \mid B_0 = (t, x_1, \dots, x_n)\}.$$

B is Brownian motion on \mathbf{R}^{n+1} , and τ is the first time that B hits the boundary corresponding to $t = 0$.

In the study of H^p spaces, Burkholder, Gundy and Silverstein [2] have obtained striking results from the probabilistic method, while the landmark paper of Fefferman and Stein [3] uses analytic methods based on kernels. P.A. Meyer [4], inspired by these papers, developed a probabilistic H^p theory for a wide class of processes and operators, but was unable to show that the various spaces he defined were identical to the classical H^p spaces. As a test case, he proposed studying functions u satisfying the heat equation $u_t = \frac{1}{2}u_{xx}$; the corresponding process would be the spacetime Brownian motion $X_t = (T - t, B_t^1, \dots, B_t^n)$.

We answer Meyer's question by showing that for the spacetime motion, the probabilistic and classical H^p spaces are identical. (The proof is not completely probabilistic as Meyer might have hoped, however.) We go on to show that for a wide class of operators and processes, the H^p spaces so defined are identical. A main feature of our work is the ability to handle degenerate operators which are like the Laplacian in some regions and like the heat operator in others. Our main tools are the Cameron-Martin formula for Brownian motion, and ideas from