

Variational Problems with Critical Sobolev Growth and Positive Dirichlet Data

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1. Introduction. In this paper, we consider the Dirichlet problem for the conformally invariant model problem of critical Sobolev growth:

$$(1.1) \quad \begin{aligned} -\Delta u &= \lambda u^{(n+2)/(n-2)} && \text{in } \Omega \\ u &= \varphi && \text{on } \partial\Omega \\ u &\geq 0 \end{aligned}$$

Problem (1.1) is formally the Euler-Lagrange equations for the variational problem

$$(1.2) \quad \int_{\Omega} |\nabla u|^2 \rightarrow \min$$

for u in the admissible class

$$\mathcal{A} = \left\{ u \in H^1(\Omega) : u - h \in H_0^1(\Omega), \int_{\Omega} u^{2n/(n-2)} = \gamma \right\},$$

where h is the harmonic extension of $\varphi \geq 0$.

It is well known that for $\varphi \equiv 0$, the existence of nontrivial solutions is often a very subtle question. For example, if Ω is starshaped there is no solution of (1.1) while if Ω has “nontrivial topology” then there do exist solutions [1].

The purpose of this paper is to show that for positive $C^{1+\beta}$ data φ , Problem (1.2) is well posed for any C^2 domain Ω . More precisely, we have the following

Theorem 1.1. *Let $\varphi \in C^{1+\beta}(\partial\Omega) \geq 0$ be positive somewhere, $\partial\Omega \in C^2$. Assume that $\gamma > \int_{\Omega} h^{2n/(n-2)}$ where h is the harmonic extension of φ . Then there is a positive minimizer $u \in C^2(\Omega) \cap C^{1+\beta}(\overline{\Omega})$ of Problem (1.2) satisfying (1.1) for a positive constant $\lambda > 0$.*