

# *A Counter-Example to the Singular Sequence Problem*

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ABSTRACT. If  $A$  and  $B$  are bounded self-adjoint operators having the same singular sequences, does it then follow that  $B - A$  is compact? We present a counter-example to this question and give an affirmative answer if  $A$  and  $B$  commute.

**Introduction.** The main purpose of this note is to give a counter-example to the *singular sequence problem*:

Let  $A$  and  $B$  be bounded self-adjoint operators in a Hilbert space  $H$ , and assume that  $A$  and  $B$  have the same singular sequences. Is it then possible to conclude that  $B - A$  is compact?

The singular sequence problem was formulated jointly by Karl Gustafson and Konrad Jörgens in 1970. The problem belongs to the context of Weyl's theorem on the invariance of the essential spectrum ([7]; cf. [1, Section 94, Satz 1], [6, Satz 9.9]) and von Neumann's converse of Weyl's theorem ([3]; cf. [1, Section 94, Satz 3]). For more motivation and discussion we refer to [2] and [5]. Also, for the case that  $\sigma_e(A)$  is countable, an affirmative answer was given in [5].

In Section 1 we present our counter-example.

In Section 2 we give some comments, and also we give an affirmative answer to the singular sequence problem for the case that  $A$  and  $B$  commute.

We conclude this introduction by some definitions and basic facts. Let  $A$  be a bounded self-adjoint operator in a Hilbert space  $H$ . The *essential spectrum*  $\sigma_e(A)$  is the set of points of  $\sigma(A)$  which are not isolated eigenvalues of finite multiplicity. A number  $\lambda \in \mathbf{R}$  belongs to  $\sigma_e(A)$  if and only if there exists a *singular sequence for  $A$  and  $\lambda$* , i.e., a sequence  $(f_n)$  in  $H$  such that  $\liminf \|f_n\| > 0$ ,  $f_n \xrightarrow{w} 0$ , and  $(A - \lambda)f_n \rightarrow 0$  ([7]; cf. [6, Satz 7.24]). Let  $B$  also be a bounded self-adjoint operator in  $H$ . If for any  $\lambda \in \sigma_e(A)$  and any singular sequence  $(f_n)$  for  $A$  and  $\lambda$ ,  $(f_n)$  is also a singular sequence for  $B$  and  $\lambda$ , then we say that  $\sigma_e(A)$  is contained in  $\sigma_e(B)$  in the sense of singular sequences, abbreviated  $\sigma_e(A) \stackrel{s}{\subset} \sigma_e(B)$ . If  $\sigma_e(A) \stackrel{s}{\subset} \sigma_e(B)$  and  $\sigma_e(B) \stackrel{s}{\subset} \sigma_e(A)$ , we say that  $A$  and  $B$  have the same singular sequences,  $\sigma_e(A) \stackrel{s}{=} \sigma_e(B)$ .