

On a Theorem of Kempf and Ness

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Let G be a complex reductive algebraic group and K a maximal compact subgroup of G . Consider a finite dimensional complex linear representation V of G . Let N be the square of the norm function obtained from a K -invariant Hermitian metric on V . Kempf and Ness prove in [5] that the function N , restricted to G -orbits in V , has remarkable properties. Namely, if O is such an orbit, then any critical point of the restriction of N to O is a point where this function assumes its absolute minimum, and the critical set, if it is non-empty, is a K -orbit and is a non-degenerate critical submanifold in the sense of Bott [2]. They prove this by analysing special functions on \mathbf{R}^n of the type $\sum e^{H_i(x)}$ ($x \in \mathbf{R}^n$), where the H_i are affine functions, apply this analysis to prove their result for an algebraic torus, and finally reduce the general problem to that for a torus, using standard facts about the Cartan decomposition of reductive groups [4, 9].

Our observation is that the function N is a strictly plurisubharmonic function on V and remains so when restricted to any complex submanifold of V ; hence the result of Kempf and Ness should hold for any K -invariant strictly plurisubharmonic function defined on a G -orbit. The principal aim of this paper is to prove that this is indeed so. Our proof is different from that of Kempf-Ness and reduces to the observation that a plurisubharmonic function on the complex plane which depends only on one variable must be convex. We then draw some immediate consequences regarding holomorphy hulls of K -invariant domains in complexifications of K -orbits. We also give a description of those homogeneous spaces of G which have at least one totally real K -orbit and use this to give another proof of Matsushima's characterization of Stein spaces of G [7, 1]. Here it should be mentioned that the principal interest of Kempf-Ness in [5] is to give criteria for closedness of G -orbits.

Proposition. *Let H be a closed complex subgroup of G and φ a K -invariant strictly plurisubharmonic function on G/H . If the critical set of φ is non-empty, then it is a K -orbit and any critical point of φ is a point where φ assumes its absolute minimum.*

The proof depends on the following lemmas which are of independent interest.