

Orthogonal Projections, Riesz Capacities, and Minkowski Content

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ABSTRACT. Let $G(n, m)$ be the Grassmannian manifold of m -dimensional linear subspaces of \mathbf{R}^n , let $\gamma_{n, m}$ be the orthogonally invariant probability measure on $G(n, m)$, and for $V \in G(n, m)$ let $P_V : \mathbf{R}^n \rightarrow V$ denote the orthogonal projection. We shall show, for example, that for any Borel set $E \subset G(n, m)$

$$\int \mathcal{H}^m(P_V E) d\gamma_{n, m} \geq c C_m(E)$$

where \mathcal{H}^m is the m -dimensional Hausdorff measure and C_m the capacity corresponding to the Riesz kernel $|x|^{-m}$. In the case $m = n - 2 \geq 1$ this gives for the Lebesgue measure \mathcal{L}^n the inequality

$$\int \mathcal{H}^{n-2}(P_V E) d\gamma_{n, n-2} V \geq c \mathcal{L}^n(E)^{(n-2)/n}$$

with the best possible constant c . We also study relations between Hausdorff measures and upper Minkowski contents of the projections.

1. Introduction. For simplicity in this introduction, I shall consider only projections of plane sets on lines, although the results of the paper will be established for projections of sets in \mathbf{R}^n on m -dimensional subspaces. For $\vartheta \in [0, \pi)$ let L_ϑ be the line through the origin in \mathbf{R}^2 forming an angle ϑ with the positive x -axis, and let $p_\vartheta : \mathbf{R}^2 \rightarrow L_\vartheta$ be the orthogonal projection. Let $E \subset \mathbf{R}^2$ be a Borel set. Our basic estimate is (see Theorem 3.2)

$$(1.1) \quad \pi^2 \left(\iint |x - y|^{-1} d\mu x d\mu y \right)^{-1} \leq \int_0^\pi |p_\vartheta(E)| d\vartheta$$