

Polynomial Approximation in Sobolev Spaces

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1. Introduction. If $f(x)$ and its derivatives of order less than or equal to ℓ are in $L^p(R^n)$, then $f(x)$ is in $L^q(R^n)$, where $\frac{1}{q} = \frac{1}{p} - \frac{\ell}{n} > 0$. In fact, we have

$$\|f\|_q \leq C \cdot \sum_{k=0}^{\ell} \|\nabla^k f\|_p.$$

This is Sobolev's inequality. It also applies to certain bounded domains D in R^n . This paper examines to what extent the derivatives of order $k < \ell$ can be omitted from the sum, with additional requirements on f . These modifications must at least rule out constant functions, of course.

For example, if we assume that $B \subset D$, where D is a convex bounded subset of R^n , and that $f|_B = 0$, then

$$\|f\|_q \leq C \cdot \|\nabla^\ell f\|_p.$$

The constant C depends on p, q, n, B , and D . We will estimate its dependence on $|B|$ and on $d = \text{diam}(D)$, using techniques similar to both the level curve approach (see Osserman [9]), and the use of Riesz potentials (see Stein [12]). A second basic question arises; if $A, B \subset D$, a bounded convex set, and $f|_B = 0$, and $f|_A = 1$, then what is a lower bound for $\|\nabla^\ell f\|_p$? The dual problem is to find an upper bound for $\|M_{AB}(z)\|_{p'}$ where $\frac{1}{p} + \frac{1}{p'} = 1$ and $M_{AB}(z)$ as defined in Section 2 is, roughly speaking, the measure of the line segments \overline{ab} , $a \in A, b \in B$ which contain z . This function is the natural generalization of the Riesz potentials for the problems at hand.

Theorem 2.3 answers this dual question. It provides the crucial estimates needed in the proof of the main theorem, Theorem 3.1, which is a version of the Bramble-Hilbert lemma about polynomial approximation in Sobolev spaces. This work originated in an attempt to find the largest possible exponent of $|B|$ in that result, and the one given there is not simple, but is indeed sharp.