

On Lebesgue Space Estimates for the Wave Equation

JØRGEN HARMSE

ABSTRACT. It is shown that the energy estimate is a special property of L^2 : similar estimates involving exponents $p \neq 2$ are false. Strichartz's global Lebesgue space estimate $\|E_+ * f\|_q \leq C\|f\|_p$ (E_+ the forward fundamental solution) is extended off the line of duality and the precise range in which this and similar solution operators are bounded given. Additional arguments yield a global estimate $\|\varphi\|_2 \leq C\|\square\varphi\|_p$ for φ compactly supported even though these solution operators do not map L^p to L^2 . Some nearly sharp local estimates (not immediate corollaries of the global bounds) are obtained. There is also a discussion of how much a solution of the wave equation can concentrate.

Introduction. It is well known that for $\varphi \in C_c^\infty((-1,1) \times B^n)$, $\|\partial_0\varphi\|_2 \leq \|\square\varphi\|_2$. On the other hand Wainger [26] showed that unless $p < \frac{2n}{n-1}$ (or $n = 1$) there is no estimate for $\|\partial_0\varphi\|_p$ in terms of $\|\square\varphi\|_p$ valid for all such φ . (It is easy to see that for $1 \leq p < q \leq \infty$ there is no estimate for $\|\partial_0\varphi\|_q$ in terms of $\|\square\varphi\|_p$ valid for $\varphi \in C_c^\infty((-1,1) \times B^n)$ and it follows by dilation that there are no values of p and q in $[1,\infty]$ for which there is an estimate for $\|\partial_0\varphi\|_q$ in terms of $\|\square\varphi\|_p$ valid for $\varphi \in C_c^\infty(\mathbf{R} \times \mathbf{R}^n)$.) Theorem 1.1. shows that there is no such estimate for $2 < p \leq \infty$. Tomas and Kenig [11] proved that certain L^∞ Fourier multipliers are unbounded on L^p for $p \neq 2$, but the relevant multiplier here is not even in L^∞ . The fact that the energy estimate nevertheless holds shows a difference between local and global problems. In addition this paper will show that disproving operator bounds does not settle questions about *a priori* estimates and that these questions need not be self-dual. Thus the statement for $1 \leq p < 2$ (Corollary 1.15) really is weaker than that for $2 < p \leq \infty$. One attempt at a stronger result raises Question 1.16: Is there a limit to the concentration of a solution of the wave equation? Propositions 1.19. and 1.20.