

K-theory Relative to a Semifinite Factor

JOHN PHILLIPS

ABSTRACT. Let M be a II_1 factor, let $N = M \bar{\otimes} \mathcal{B}(H)$ be the corresponding II_∞ factor, and let \mathcal{K}_N be the closed two-sided ideal in N generated by the elements of finite trace. For an arbitrary C^* -algebra A , we define the abelian groups, $K_0^N(A)$ and $K_1^N(A)$. $K_0^N(A)$ is, roughly speaking, the “groupification” of the equivalence classes of projections in $A \otimes \mathcal{K}_N$, while $K_1^N(A)$ is the group of connected components of invertible operators in $(A \otimes \mathcal{K}_N)^1$. Our main result states that $K_*^N(A)$ and $K_*(A \otimes M)$ are naturally isomorphic and thus K_*^N is a stable, half-exact, homotopy functor satisfying Bott periodicity. This generalizes (with completely different proofs) the results of M. Breuer, who had defined the functor K_*^N for commutative C^* -algebras using the idea of vector bundles relative to \mathcal{K}_N .

We also exhibit a B. D. F. pairing:

$$K_1^N(A) \otimes_{\mathbf{Z}} \text{Ext}^N(A)^{-1} \rightarrow \mathbf{R},$$

where $\text{Ext}^N(A)^{-1}$ is the group of invertible extensions of A by \mathcal{K}_N defined by P. Fillmore. For a large class of algebras, A , this pairing yields a “universal coefficient theorem”, that is, an isomorphism of \mathbf{R} -vector spaces:

$$\text{Ext}^N(A)^{-1} \cong \text{Hom}_{\mathbf{R}}[K_1^N(A), \mathbf{R}].$$

This is, of course, closely related to the universal coefficient theorem of J. Rosenberg and C. Schochet.