

High Order Regularity for Conservation Laws

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ABSTRACT. We study the regularity of discontinuous entropy solutions to scalar hyperbolic conservation laws with uniformly convex fluxes posed as initial value problems on \mathbb{R} . For positive α we show that if the initial data has bounded variation and the flux is smooth enough then the solution $u(\cdot, t)$ is in the Besov space $B_\sigma^\alpha(L^\sigma)$ where $\sigma = 1/(\alpha + 1)$ whenever the initial data is in this space. As a corollary, we show that discontinuous solutions of conservation laws have enough regularity to be approximated well by moving-grid finite element methods. Techniques from approximation theory are the basis for our analysis.

1. Introduction. Discontinuities may form in the solution $u(x, t)$ of the hyperbolic conservation law

$$(C) \quad \begin{aligned} u_t + f(u)_x &= 0, & x \in \mathbb{R}, \quad t > 0, \\ u(x, 0) &= u_0(x), & x \in \mathbb{R}, \end{aligned}$$

even if the flux f and the initial data u_0 are smooth. Hence, classical solutions of (C) do not generally exist. Weak solutions of (C) are not unique, but both existence and uniqueness of weak solutions that satisfy one of several auxiliary “entropy” condition were shown by Oleinik [21], Vol’pert [26], and Kružkov [14]. The regularity of these weak solutions is the topic of this paper. Namely, we are interested in smoothness spaces X that are regularity spaces for (C), i.e., $u_0 \in X$ implies that $u(\cdot, t) \in X$ for all positive t .

The Sobolev spaces $W^{\alpha, p}$ for $p \geq 1$ and $\alpha \geq 1$ contain only continuous functions, and therefore are not appropriate candidates for X . More generally, a Sobolev-type embedding theorem implies that if $\alpha p > 1$ then functions in the Besov spaces $B_q^\alpha(L^p)$ (q is a secondary index of smoothness; see Section 3) are again continuous. Consequently, if one desires high order smoothness ($\alpha > 1$) one must measure smoothness in L^p spaces with $0 < p \leq \frac{1}{\alpha} < 1$. Such Besov