

Heat Kernel on Noncomplete Manifolds

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ABSTRACT. Let M be a smooth manifolds and L a second order, nondegenerate elliptic operator on M . Let $p(t,x,y)$ be the heat kernel on M associated with the operator L . The basic asymptotic relation $\lim_{t \rightarrow 0} t \log p(t,x,y) = -d(x,y)^2/2$ (d is the Riemannian distance on M determined by L) may fail for noncomplete manifolds. We prove that the best condition under which this relation holds is $d(x,y) \leq d(x,\infty) + d(y,\infty)$.

1. Introduction. Let M be a smooth manifold and L a second order, non-degenerate elliptic operator on M . The operator L determines a Riemannian metric d on M . Let Δ be the Laplace-Beltrami operator determined by the Riemannian metric. Then L can be written in the form $L = \frac{1}{2}\Delta + F$ with a smooth vector field F . The minimal heat kernel on M associated with the elliptic operator L is denoted by $p(t,x,y)$. Take any two points x, y on M . We are concerned with the behavior of $p(t,x,y)$ as $t \rightarrow 0$. The basic asymptotic result was proved by Varadhan [6]:

$$(1.1) \quad \lim_{t \rightarrow 0} t \log p(t,x,y) = -\frac{1}{2} d(x,y)^2.$$

This result and the subsequent extension to hypoelliptic operators (Léandre [5]) holds under the assumption that manifold M is complete in the Riemannian metric determined by the operator. The completeness of M is often implied by assumptions on the operators involved. However, it is sometimes neglected by some authors. It was first observed in Azencott et al [1] that the basic asymptotic relation (1.1) may fail for noncomplete manifolds. In the same work it was proved that (1.1) does hold if the condition

$$(1.2) \quad d(x,y) \leq \max\{d(x,\infty), d(y,\infty)\}$$

is satisfied. Here $d(x,\infty)$ is the distance of x to the “infinity” defined by

$$(1.3) \quad d(x,\infty) = \sup_{\substack{K \subset M \\ K \text{ compact}}} d(x, K^c).$$