

The Duals of Lipschitz Spaces Defined on Closed Sets

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ABSTRACT. It is known that the second dual of the “small” Lipschitz space $\lambda_\alpha(\mathbf{R}^n)$ is the Lipschitz space $\Lambda_\alpha(\mathbf{R}^n)$, and that the dual of $\lambda_\alpha(\mathbf{R}^n)$ can be identified as a certain Besov space. The purpose of this paper is to generalize these results to Lipschitz spaces $\lambda_\alpha(F)$ and $\Lambda_\alpha(F)$ of functions defined on a closed subset F of \mathbf{R}^n . The dual of $\lambda_\alpha(F)$ is characterized as a space defined by means of atomic sums, where in the definition of an atom moments are taken with respect to a measure with support F satisfying a doubling condition.

0. Introduction. Let F be a closed subset of \mathbf{R}^n , $0 < \alpha < 1$, and let $\Lambda_\alpha(F)$ denote the Lipschitz space of order α on F , i.e. $f \in \Lambda_\alpha(F)$ if and only if $|f(x) - f(y)| \leq M|x - y|^\alpha$, $x, y \in F$ and $|f(x)| \leq M$, $x \in F$, for some constant M . The norm of f in $\Lambda_\alpha(F)$ is the infimum of the possible constants M . Let furthermore $\lambda_\alpha(F)$ denote the closure of $C_0^\infty(F)$ in $\Lambda_\alpha(F)$. In this paper we define a certain space $S_\alpha(F)$ and prove that the dual of $\lambda_\alpha(F)$ is $S_\alpha(F)$ and that the dual of $S_\alpha(F)$ is $\Lambda_\alpha(F)$. Furthermore, we obtain analogous results for $\alpha > 0$, but we then have to impose further restrictions on F assuming that F preserves Markov’s inequality (see Section 1).

The spaces $S_\alpha(F)$ are defined by means of atomic sums. In the definition of an atom, it is required that a number of moments of the atoms are zero, moments with respect to a measure μ with support F satisfying a doubling condition. Such measures always exist, due to a result in Volberg–Konyagin [8].

It was proved in de Leeuw [6], that the second dual of $\lambda_\alpha(T)$ is $\Lambda_\alpha(T)$, $0 < \alpha < 1$, where T is the circle. In Triebel [7] it is proved that the dual of $\lambda_\alpha(\mathbf{R}^n)$ is the Besov space $B_{1,1}^{-\alpha}(\mathbf{R}^n)$, which has the dual $\Lambda_\alpha(\mathbf{R}^n)$, and it follows from the results on decompositions of Besov spaces in \mathbf{R}^n , given in Frazier–Jawerth [2], that the space $B_{1,1}^{-\alpha}(\mathbf{R}^n)$ is the same as $S_\alpha(\mathbf{R}^n)$.