

# *Lipschitz Approximation by Harmonic Functions and Some Applications to Spectral Synthesis*

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ABSTRACT. For  $0 < s \leq 1$ , we characterize those compact sets  $X$  with the property that each function harmonic in  $\overset{\circ}{X}$  and satisfying a little  $\mathbf{o}$  Lipschitz condition of order  $s$  is the limit in the Lipschitz norm of order  $s$  of functions harmonic on neighbourhoods of  $X$ . As an application of the methods we give a spectral synthesis result in the space of locally integrable functions whose laplacian belongs to  $B^p(\mathbb{R}^d)$ , the containing Banach space of the Hardy space  $H^p(\mathbb{R}^d)$ .

**0. Introduction.** Let  $X$  be a compact subset of  $\mathbb{R}^d$  and let  $\Lambda^s(X)$ ,  $0 < s < 1$ , be the usual Banach space of Lipschitz functions of order  $s$  on  $X$ . That is,  $f \in \Lambda^s(X)$  if and only if

$$\|f\|_s = \sup\{|f(x) - f(y)| |x - y|^{-s} : x, y \in X, x \neq y\} < \infty.$$

If  $\|\cdot\|_\infty$  is the supremum norm on  $X$ ,  $\|f\|_\infty + \|f\|_s$  is a Banach space norm on  $\Lambda^s(X)$ . An important role in what follows will be played by the space  $\lambda^s(X)$  which is the set of functions  $f$  in  $\Lambda^s(X)$  satisfying  $|f(x) - f(y)| = o(|x - y|^s)$  as  $|x - y| \rightarrow 0$ . Alternatively  $\lambda^s(X)$  can be described as the closure in  $\Lambda^s(X)$  of  $C^\infty(\mathbb{R}^d)|_X$ .

We are interested in the problem of understanding the space  $H^s(X)$ , the closure in  $\Lambda^s(X)$  of  $\{f|_X : f \text{ is harmonic on some neighbourhood of } X\}$ . There are two obvious necessary conditions for  $f \in H^s(X)$ :  $f$  must be harmonic on  $\overset{\circ}{X}$ , and  $f$  must belong to  $\lambda^s(X)$ . If we set

$$h^s(X) = \lambda^s(X) \cap \{f : \Delta f = 0 \text{ on } \overset{\circ}{X}\}$$

we then have  $H^s(X) \subset h^s(X)$  and it turns out that the above inclusion can be strict. Thus the problem arises of characterizing those  $X$  for which  $H^s(X) =$