

Deforming Curves on Convex Surfaces to Simple Closed Geodesics

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ABSTRACT. This paper explores the behavior of closed curves on strictly convex surfaces which evolve according to the equation $X_t = \frac{k}{K}N$ where X is the position vector, k is the geodesic curvature, and K is the Gauss curvature of the surface at the point. Those simple closed curves which bisect Gauss area, i.e., divide the surface into two regions on which $\int K dA = 2\pi$ preserve this property under the flow given. If the initial curve is close enough to a closed geodesic then the curve flows to geodesic. If the geodesic curvature of the curve approaches infinity at some time T then a cusp must develop at time T as well.

In this paper we use a variant of the “curve shortening” evolution to find simple closed geodesics on convex surfaces as the stable minimum of a flow.

A standard technique for finding closed geodesics is to produce the curve which minimizes length in a given homotopy class and prove that it is a geodesic. In a simply connected space the only length minimizing curves are points and this technique fails.

An alternative is to use a minimax argument: Consider a collection of continuous families of curves indexed by points in S^1 so that each family forms a map from $S^1 \times S^1$ to the surface of the convex body which is of degree one. Within that collection find that family whose longest curve has the least length. This curve is a geodesic.

Another possibility, proposed by Poincaré, is to find the curve of minimum length among all simple closed curves which divide the surface into two pieces each having total Gauss curvature equal to 2π . C. Croke [C] gave a rigorous proof of the existence and embeddedness of such a minimizing curve. In this paper we show how the curve can be obtained by evolving a nearby geodesic curve according to a particular nonlinear parabolic equation.

Let γ be a simple closed curve on a closed convex surface M with strictly positive Gaussian curvature. Assume that the curve γ divides the surface into