

# *Solving the Dirichlet Problem in the Plane by Means of the Cauchy Integral*

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**1. Introduction.** In this paper, the solution to the classical Dirichlet problem in a planar domain is described in terms of the Szegő projection, giving rise to a numerical method for solving the Dirichlet problem using explicit *non-singular* integral equations of the most well behaved kind.

The idea of this paper is very simple. To illustrate it, suppose that  $\Omega$  is a bounded simply connected domain with  $C^\infty$  smooth boundary in the complex plane. The classical theory of the Dirichlet problem states that every continuous function  $\varphi$  on the boundary of  $\Omega$  is the boundary value of a harmonic function  $u$  on  $\Omega$  which extends continuously to  $\bar{\Omega}$ . Furthermore, if  $\varphi$  is  $C^\infty$  smooth on the boundary of  $\Omega$ , then  $u$  is in  $C^\infty(\bar{\Omega})$ . Every harmonic function in  $C^\infty(\bar{\Omega})$  can be written as a sum  $u = h + \bar{H}$  where  $h$  and  $H$  are holomorphic functions on  $\Omega$  which are in  $C^\infty(\bar{\Omega})$ . These holomorphic functions are uniquely determined up to additive constants. In this paper, we construct the boundary values of the holomorphic functions  $h$  and  $H$  by expressing them as Szegő projections of explicit functions. In the past, the Szegő projection was considered to be abstractly defined and difficult to evaluate numerically. However, since the work of N. Kerzman and E. M. Stein [6], the Szegő projection and kernel can be viewed concretely and are eminently computable by means of the Kerzman-Stein integral equation, which relates these objects to the Cauchy transform and kernel. The Kerzman-Stein kernel which appears in this integral equation is not singular; it is a  $C^\infty$  smooth kernel. Thus, having expressed  $h$  and  $H$  concretely, we obtain a method whereby the harmonic function which solves the Dirichlet problem can be evaluated at interior points of  $\Omega$  via the Cauchy integral formula. From a practical point of view, the method for computing  $h$  and  $H$  is very simple. The computations involve solving a Fredholm integral equation of the second kind with  $C^\infty$  smooth kernel. Furthermore, the computations are made on the boundary of  $\Omega$ , which is a one dimensional object. The extension to the two dimensional  $\Omega$  is made via the Cauchy integral. Consequently, we will exhibit a simple one dimensional method for numerically solving the Dirichlet problem which uses a non-singular integral equation of the most classical and well behaved kind.