

# On Almost Isospectral Manifolds

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**ABSTRACT.** In this note, we introduce a new notion of  $\alpha$ -isospectral manifolds. Then we prove the rigidity theorem on  $S^4$ ,  $S^5$  and  $CP^n$ . As a special case, we have the isospectral rigidity on  $CP^n$  for  $n \geq 4$ .

**1. Introduction.** Let  $(M, g)$  be a closed Riemannian manifold. The Laplace-Beltrami operator  $\Delta$  acting on functions on  $M$  has a discrete spectrum:  $\text{Spec}(M, g) : 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \nearrow \infty$ . We shall say that two Riemannian manifolds  $(M, g)$  and  $(\bar{M}, \bar{g})$  are isospectral if  $\text{Spec}(M, g) = \text{Spec}(\bar{M}, \bar{g})$ . One of the basic problems in the spectrum theory is the following: Is a Riemannian manifold determined up to isometry by its spectrum? In general, this is not true. Milnor [BGM] gave an example of two 16-dimensional flat tori which are isospectral but not isometric. This is the first example of non-isometric isospectral manifolds. Other examples have been found afterwards: 5-dimensional lens spaces and compact manifolds with constant curvature  $-1$ . However, in some cases, the spectrum do determine the manifolds. For instance,

- (1) Two 2-dimensional isospectral flat tori are isometric ([BGM]).
- (2) For  $n \leq 5$ , if  $(M, g)$  is isospectral to  $(S^n, g_{\text{can}})$  (resp.  $(RP^n, g_{\text{can}})$ ), then  $(M, g)$  is isometric to  $(S^n, g_{\text{can}})$  (resp.  $(RP^n, g_{\text{can}})$ ) ([BGM], [T1]).
- (3) For  $n \leq 6$ , a Kähler manifold  $(M, g)$  has the same spectrum as the complex projective space  $CP^n$  with the Fubini-Study metric  $g_0$ , then  $(M, g)$  is holomorphically isometric to  $(CP^n, g_0)$  ([T1]).

In this note, we study the perturbation phenomena of these cases. The Weyl's asymptotic formula says that  $\lambda_m \sim c_n \text{Vol}(M) m^{2/n}$  where  $n$  is the dimension of  $M$  and  $c_n$  is a universal constant depending only on  $n$ . It is natural to ask the following : In the above examples (1), (2) and (3), can we relax the conditions? Namely, if we perturb the metrics on  $M$  such that the eigenvalues  $\lambda_m = \lambda_m^{\text{new}}$  and  $\bar{\lambda}_m = \lambda_m^{\text{old}}$  of the Laplacian operators associated to the new and old metrics only differ by a lower term than  $m^{2/n}$ , can we still have the rigidity result? In Section 2, we shall see that if  $\lambda_m$  and  $\bar{\lambda}_m$  differ by a fixed constant, then this is not true. In Section 3, we show that for some cases, if  $\lambda_m$  and  $\bar{\lambda}_m$