

# An Application of a Parabolic Comparison Principle to Free Boundary Problems

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In this paper we obtain in a very simple manner free boundary regularity via a comparison principle for solutions of parabolic equations under the assumption that the solution is monotone in some spatial direction, thus extending the result of [2] to the parabolic case.

Although we can deal with more general situations, for simplicity, we consider the classical one-phase Stefan problem in a local setting, i.e., we are given a function  $u$  such that

- (i)  $u(x, t) \geq 0$  in  $D_R := B_R(x_0) \times (t_0 - R^2, t_0 + R^2)$ , with  $(x_0, t_0) \in \partial\{u > 0\}$ ,
- (ii)  $Hu := \Delta u - u_t = 1$  in  $D_R \cap \{u > 0\}$ ,

where  $B_R(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < R\}$ . It is well known (see [3]) that  $u \in C_x^{1,1}(D_{R-\varepsilon})$ ,  $u \in C_t^1(D_{R-\varepsilon})$  for any  $\varepsilon > 0$ . The extra assumption that in some spatial direction, say  $e_1$ ,  $u$  is monotone (i.e.,  $D_{e_1}u \geq 0$ ) may be inferred in many cases from global situations. This implies that the free boundary can be represented as a graph of a function  $x_1 = f(x_2, \dots, x_n, t)$ . By an argument due to H. W. Alt (see [1]) adapted to the parabolic case, one can show that in compact subsets of  $D_R$  there is a whole cone  $K(e_1, \vartheta) = \{\sigma \in \mathbb{R}^{n+1} : |\sigma| = 1, \sigma \cdot e_1 < \cos \vartheta\}$  of directions such that  $D_\sigma u \geq 0 \forall \sigma \in K(e_1, \vartheta)$ . Hence  $f$  is Lipschitz continuous.

We intend to prove the following:

**Theorem.** *Let  $u$  be a solution of (i) and (ii) above. Suppose that  $D_{e_1}u \geq 0$  and  $f$  is a Lipschitz function. Then  $f$  is a  $C^{1,\alpha}$  function for some  $0 < \alpha < 1$ .*

For the proof of the theorem we need a control of the gradient, as follows.

**Lemma 1.** *Let  $(0, 0) \in \partial\{u > 0\}$  and  $B_{2r}(0) \times (-4r^2, 4r^2) \subset D_R$ . Then there exist constants  $C > c > 0$  depending only on the aperture of the cone  $K(e_1, \vartheta)$ ,  $n$ ,  $\|u\|_{C_x^{1,1}}$  and  $\|u\|_{C_t^{0,1}}$  such that*

$$c\sqrt{u(x, t)} \leq |\nabla u(x, t)| \leq C\sqrt{u(x, t)}$$

$\forall (x, t) \in B_r(0) \times (-r^2, r^2)$ , where  $\nabla u = (\nabla_x u, u_t)$ .