

Concentration in Sequences of Solutions to the Nonlinear Klein-Gordon Equation

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ABSTRACT. We study the compactness of a sequence of weak solutions to the (p -th power) nonlinear Klein-Gordon equation with uniform natural energy bounds. A refinement of the Rellich's Theorem of compact imbedding is established to show that the loss of strong convergence in the $L^{p+1}(\mathbb{R}^3 \times (0, T))$ norm happens at most on a concentration set with Hausdorff dimension of $1 + \frac{p-4}{p-2}$ in space and time. And L^{p+1} strong convergence is established, provided that the concentration set can be split into two parts, whose temporal and spatial projections respectively have Hausdorff dimensions less than one, as is the case for radial solutions. It is also shown that the concentration in the associated action in the weak star topology of bounded measures is directly caused by the loss of L^{p+1} strong convergence.

Introduction. A weak solution exists for all time for the Cauchy problem for the p -th power nonlinear Klein-Gordon (NLKG) equation

$$u_{tt} - \Delta u + m^2 u + g|u|^{p-1}u = 0, \quad x \in \mathbb{R}^3, t \in \mathbb{R}^+,$$

with initial data

$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x).$$

Here $m > 0$, $g > 0$, $1 < p < \infty$, $\varphi(x) \in H^1(\mathbb{R}^3) \cap L^{p+1}(\mathbb{R}^3)$ and $\psi(x) \in L^2(\mathbb{R}^3)$ (see [15]). A function $u(x, t) \in L^\infty((0, T), H^1(\mathbb{R}_x^3)) \cap L^{p+1}((0, T) \times \mathbb{R}_x^3)$ with $u_t(x, t) \in L^\infty((0, T), L^2(\mathbb{R}_x^3))$ is a weak solution of the NLKG equation if

$$\int_0^T \int_{\mathbb{R}^3} (u_t \xi_t - \nabla_x u \cdot \nabla_x \xi - m^2 u \xi - g|u|^{p-1}u \xi) dx dt = 0$$

for all test functions $\xi(x, t) \in H_0^1 \cap L^{p+1}((0, T) \times \mathbb{R}^3)$, and for all $T > 0$ (cf. J. L. Lions [12]).