

# A Fixed Point Property for the Lorentz Space $L_{p,1}(\mu)$

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ABSTRACT. Let  $(X, \Sigma, \mu)$  be a  $\sigma$ -finite measure space. Our main result is that every non-expansive mapping from a weak-star compact convex subset of  $L_{p,1}(\mu)$  into itself has a fixed point. We prove this by showing that  $L_{p,1}(\mu)$  has the weak-star uniform Kadec-Klee property.

**1. Introduction.** Let  $(X, \Sigma, \mu)$  be a  $\sigma$ -finite measure space. For a  $\mu$ -measurable function  $f$  we define the distribution of  $f$  by  $d_f(t) = \mu(\{x : |f(x)| > t\})$ ,  $0 < t < \infty$ , and the decreasing rearrangement of  $f$  by  $f^*(t) = \inf\{s > 0 : d_f(s) \leq t\}$ . For  $1 < p < \infty$ , the Lorentz space  $L_{p,1}(\mu)$  is the Banach space of all (equivalence classes of)  $\mu$ -measurable functions  $f$  under the norm  $\|f\| = \int_0^\infty f^*(t) d(t^{1/p})$ . (Throughout, we shall always take  $1 < p < \infty$ .)  $L_{p,1}(\mu)$  is a dual space whose predual is the closure in  $L_{p,1}(\mu)^* = L_{p',\infty}(\mu)$  (where  $\frac{1}{p} + \frac{1}{p'} = 1$ ) of the  $\mu$ -integrable simple functions, under the usual pairing  $(f, g) = \int_X fg d\mu$ .

Let  $C$  be a closed bounded convex subset of a Banach space  $(E, \|\cdot\|)$ . A map  $T : C \rightarrow C$  is said to be *nonexpansive* if  $\|Tx - Ty\| \leq \|x - y\|$  for all  $x, y \in C$ . The main result of the paper is that  $L_{p,1}(\mu)$  enjoys the weak-star fixed point property.

**Theorem.** *Let  $C$  be a weak-star compact convex subset of  $L_{p,1}(\mu)$ . Then, every nonexpansive mapping on  $C$  has a fixed point.*

In [4] van Dulst and Sims proved that a dual Banach space with a weak-star sequentially compact unit ball enjoys the weak-star fixed point property, provided that a certain convexity condition is satisfied. To be precise, a dual space  $E$  has the *weak-star uniform Kadec-Klee property* if the following holds: given  $\varepsilon > 0$ , there is a  $\delta(\varepsilon) > 0$  such that for every weak-star convergent sequence  $(f_n)$  with  $\|f_n\| = 1$ ,  $\|f_n - f_m\| \geq \varepsilon$  ( $m \neq n$ ), and with weak-star limit  $f$ , we