

Uniqueness of Energy Minimizing Maps for Almost All Smooth Boundary Data

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ABSTRACT. Here for a smooth bounded Euclidean domain Ω and a smooth Riemannian manifold N , we show that almost every smooth map $\varphi : \partial\Omega \rightarrow N$ serves as boundary data for at most one energy minimizing map $u : \Omega \rightarrow N$ (Theorem 5.2).

We also obtain some uniform boundary regularity estimates for energy minimizers (Theorem 2.1), which not only are important to our proof, but also imply some other properties of energy minimizers (Corollaries 2.6–2.9).

1. Introduction. We suppose that Ω is a bounded $C^{3,\alpha}$ domain in \mathbf{R}^m and N is a $C^{4,\alpha}$ compact Riemannian manifold without boundary, and consider φ in $C^{2,\alpha+}(\partial\Omega, N)$, the $\|\cdot\|_{C^{2,\alpha}}$ closure of $C^3(\partial\Omega, N)$ in $C^{2,\alpha}(\partial\Omega, N)$. We construct a finite measure μ on $C^{2,\alpha+}(\partial\Omega, N)$ which has positive value on each nonempty open subset of $C^{2,\alpha+}(\partial\Omega, N)$, and prove that

For μ , $\frac{1}{N}$, almost all $\varphi \in C^{2,\alpha+}(\partial\Omega, N)$, there exists at most one minimizer of the energy $\int_{\Omega} |\nabla u|^2 dx$ among $\{u \in L^{1,2}(\Omega, N) : u|_{\partial\Omega} = \varphi\}$.

So in particular, the nonuniqueness set Z consisting of all $\varphi \in C^{2,\alpha+}(\partial\Omega, N)$ each of which serves as boundary data of at least two different energy minimizers is of first category.

It is well known that if $u_1, u_2 \in L^{1,2}(\Omega, N)$ are energy minimizers with same boundary data, and if N is of nonpositive sectional curvature and u_1 and u_2 are homotopic, or if the images of u_1 and u_2 are contained in a same geodesic ball, then $u_1 = u_2$. See [HR], [SR, Theorem 2.10] and [JK]. Without such assumptions, F. Almgren and E. Lieb [AL, Theorem 4.1] proved that the boundary data having unique energy minimizers are actually dense in $H^1(\partial\Omega, N)$.

But in general, energy minimizers do not have uniqueness. For example, R. Hardt and F. H. Lin in [HL2] gave a $\varphi \in C^{2,\alpha}(\mathbf{S}^2, \mathbf{S}^2)$, which serves as boundary data of two energy minimizers $u_1, u_2 \in L^{1,2}(\mathbf{B}^3, \mathbf{S}^2)$, with u_1 being smooth while u_2 being singular. (See also [AL].)