

Geometric Integration on Fractal Curves in the Plane

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ABSTRACT. We provide a method of integrating 1-forms over curves with zero area which need not have any tangent vectors, e.g. fractal curves. For Lipschitz curves this integral agrees with the usual line integral of 1-forms. The minimal requirement is that the box dimension of the curve should be smaller than one plus the Hölder exponent satisfied by the form; when this condition is violated there are classical obstructions.

We study the continuity of the integral by introducing a new norm (the d -flat norm) on the space of polyhedral 1-chains, well-adapted to Hölder forms. We employ the abstract framework of Whitney's flat chains and cochains in this new context to obtain a sharper theory that reveals more clearly the nature of the obstruction Whitney discovered in the 30's.

1. Introduction. In 1935 H. Whitney constructed a C^1 function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ that was “not constant on a connected set of critical points” [18]. This example has been very influential, stimulating work in several directions, e.g. [2], [11], [12], [15], [19].

One of its more important roles is as a counterexample in integration theory, in the following way. The function f is constructed so that $\nabla f \equiv 0$ on an arc γ , even though f is increasing along γ . By considering the continuous 1-form $\omega = df$, this shows *there can be no generalized integral defined over all continuous arcs γ and 1-forms ω that satisfies both*

$$(1.1) \quad \int_{\gamma} \omega = f(\gamma(1)) - f(\gamma(0)) \quad \text{if } \omega = df,$$

and

$$(1.2) \quad \omega|_{\gamma} = 0 \text{ implies } \int_{\gamma} \omega = 0.$$