

# Volume Maximization in Semi-Riemannian Manifolds

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**Introduction.** The purpose of this paper is to report on the extension of Harvey and Lawson's notion of a calibration on a riemannian manifold to the semi-riemannian category. Recall from [HL] that a calibration on an  $n$ -dimensional riemannian manifold  $X$  is a closed exterior  $k$ -form  $\varphi$  satisfying

$$\varphi|_{\xi} \leq \text{vol}_{\xi}$$

for all oriented tangent  $k$ -planes  $\xi$  to  $X$ . The fundamental lemma of calibrations (FLC) then states that if  $M$  is a compact oriented  $k$ -dimensional submanifold of  $X$  satisfying  $\varphi|_M = \text{vol}_M$ , then  $M$  is homologically volume minimizing in  $X$ ; that is,  $\text{vol}(M) \leq \text{vol}(M')$  for any  $M'$  with  $\partial M' = \partial M$  and  $[M - M'] = 0$  in  $H_k(X; \mathbf{R})$ . See [HL] for the now standard proof. In the semi-Riemannian category, say, on a manifold  $X^{p,q}$  (with metric of signature  $(p, q)$ ), we will show that there exist closed  $p$ -forms  $\varphi$  which satisfy  $\varphi|_{\xi} \geq \text{vol}_{\xi}$  for those oriented tangent planes to  $X$  which are contained in certain naturally defined connected subsets of the full set of oriented tangent  $p$ -planes  $X$ . (See Section I.) Because of the direction of the inequality, any submanifold  $M \subset X$  satisfying  $\varphi|_{\xi} = \text{vol}_{\xi}$  is homologically volume maximizing among a naturally restricted class of submanifolds. The proof is the same as in the positive definite category. (This fact will also be referred to as the FLC.)

Section I further discusses the general case in the semi-Riemannian category, as well as the motivating example, due to Harvey, and some initial results. In Section II, we show that the now classical examples of Harvey and Lawson [HL] can be "dualized" to the signature case, yielding large classes of volume maximizing submanifolds of various  $\mathbf{R}^{p,q}$ . These geometries include complex geometry on  $\mathbf{C}^{p,q}$ , special lagrangian geometry on  $\mathbf{R}^{n,n}$ , and some exceptional geometries associated to the split octonians. In Section III, some of the differences between the positive definite and signature cases are emphasized, and some open problems discussed.

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