

Non Uniqueness of Minimal Graphs

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0. Introduction. The weak form of the Plateau problem for minimal graphs on an open, connected, and bounded subset Ω of \mathbf{R}^n consists in minimizing the functional

$$F_\varphi(u) = \int_{\Omega} \sqrt{1 + |Du|^2} \, dx + \int_{\partial\Omega} |u - \varphi| \, d\mathcal{H}^{n-1}$$

among all functions $u : \bar{\Omega} \rightarrow \mathbf{R}$ of a suitable class in which F_φ makes sense. The function $\varphi \in L^1(\partial\Omega)$ gives the prescribed boundary values for u . The class of the competing functions u is usually $BV(\Omega)$. Indeed, it is possible to define the area $\int_{\Omega} \sqrt{1 + |Du|^2}$ of the graph of u for every $u \in BV(\Omega)$, and furthermore each function $u \in BV(\Omega)$ has a trace on $\partial\Omega$ which belongs to $L^1(\partial\Omega)$. It is well known that a minimizer $u \in BV(\Omega)$ of F_φ exists: it is actually smooth within Ω and fulfils the minimal surfaces equation

$$\operatorname{div} \left(\frac{Du}{\sqrt{1 + |Du|^2}} \right) = 0.$$

As for the boundary values, we quote the following important theorem of M. Miranda [M].

Theorem. *If φ is continuous in a point $x_0 \in \partial\Omega$ and $\partial\Omega$ is of class C^2 and strictly convex in a neighborhood of x_0 , then the minimizer u of F_φ attains at x_0 the value $\varphi(x_0)$.*

Note that

$$F_\varphi(\lambda u + (1 - \lambda)v) \leq \lambda F_\varphi(u) + (1 - \lambda)F_\varphi(v) \\ \forall u, v \in BV(\Omega), \forall \lambda \in [0, 1]$$

and the equality holds only if $Du = Dv$ in Ω . Hence two minimizers differ by a constant, and the above theorem ensures the uniqueness of the minimizers provided that φ has at least one continuity point within a convex part of $\partial\Omega$.