

Mappings and Morse Decompositions in the Conley Index Theory

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ABSTRACT. Flow maps were introduced to the Conley index theory by the author. In this paper, they are related to the index theory for Morse decompositions. Flow maps commute with the topological structures of the index and with homology attractor-sequences, but do not always commute with connection matrices. A condition for commutativity is developed, and the dynamical significance of both commutativity and its failure are described.

1. Introduction. The Conley index is a tool for the study of compact invariant sets in non-compact flows. It is able to describe both the global properties of such a set and the internal structure of the flow on the set. One such internal structure is a Morse decomposition—a generalization of the unstable manifold decomposition of a gradient flow. A Morse decomposition of an invariant set S is a collection of invariant subsets (Morse sets) with a partial order on them such that their complement in S consists of orbits which run from one Morse set to another (connecting orbits) and which respect the partial order. The Conley index theory provides techniques by which Morse sets and connecting orbits can be identified and described.

This is done in two steps, one topological and one algebraic. First, an index pair is associated to S ([1], [12]), and an index filtration to the Morse decomposition ([2], [3]). These are collections of spaces chosen to capture the behavior of the flow in a neighborhood of the invariant sets. From these, the Conley indices of the sets are derived as homotopy types of quotient spaces formed from the index spaces. The homology of these homotopy types defines the homology Conley index. The Morse inequalities [2], attractor-repeller sequences [6] and connection matrices [4] generate algebraic relations on the homology indices of the elements of a Morse decomposition. When translated back to the topology, these relations provide information about the structure of the Morse sets and connecting orbits.