

Quasiregular Mappings with Finite Multiplicity and Royden Algebras

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Introduction. In the study of quasiconformal mappings, one often wishes to know if two domains Ω and Ω' in Euclidean n -space are quasiconformally equivalent; that is, if there exists a quasiconformal mapping $f : \Omega \rightarrow \Omega'$. This problem is extremely difficult even when $n = 2$. However, one implicit characterization of quasiconformally equivalent domains Ω and Ω' is that $A(\Omega)$ and $A(\Omega')$, their respective Royden algebras, are algebraically isomorphic. It has been shown that a quasiconformal mapping $f : \Omega \rightarrow \Omega'$ always induces an algebra isomorphism between $A(\Omega')$ and $A(\Omega)$. Conversely, an algebra isomorphism $T : A(\Omega') \rightarrow A(\Omega)$ always induces a quasiconformal mapping of Ω onto Ω' . These theorems were proven for the two-dimensional case by M. Nakai in [N], then generalized for n dimensions by L. Lewis in [Le]. J. Lelong-Ferrand formulated corresponding theorems for n -dimensional Riemannian manifolds [L-F]. Lelong-Ferrand also expanded the theory in part to apply to quasiregular mappings. Her work shows that a quasiregular mapping $f : \Omega \rightarrow \Omega'$ with finite multiplicity induces an algebra homomorphism $T : A(\Omega') \rightarrow A(\Omega)$.

In this paper we build on the theory developed in [L-F] and [Le] to give a new characterization of domains Ω and Ω' for which there exists a quasiregular mapping $f : \Omega \rightarrow \Omega'$ with finite multiplicity. We show that A , the image of $A(\Omega')$ under T , is a subalgebra of $A(\Omega)$ which satisfies certain desirable conditions on the separation of sets in Ω by functions in A . We develop the Royden compactification of a subalgebra of $A(\Omega)$ and use this development to show that if A is a subalgebra of $A(\Omega)$ which satisfies the desired separation conditions, then an algebra isomorphism $T : A(\Omega') \rightarrow A$ induces a quasiregular mapping $f : \Omega \rightarrow \Omega'$ whose multiplicity is bounded by a value depending only on the norm of T . We thus characterize quasiregular mappings with finite multiplicity in terms of isomorphisms between Royden algebras and subalgebras. We similarly characterize closed quasiregular mappings and provide examples which illustrate the necessity of the given conditions on A for obtaining these characterizations.