

A Sharp Pointwise Bound for Functions with L^2 -Laplacians and Zero Boundary Values on Arbitrary Three-Dimensional Domains

WENZHENG XIE

1. Introduction. In this paper we establish the following:

Theorem. *Let Ω be an arbitrary open set in \mathbb{R}^3 . If $u \in \hat{H}_0^1(\Omega)$ and $\Delta u \in L^2(\Omega)$, then*

$$(1) \quad \sup_{\Omega} |u| \leq \frac{1}{\sqrt{2\pi}} \|\nabla u\|^{1/2} \|\Delta u\|^{1/2}.$$

The constant $\frac{1}{\sqrt{2\pi}}$ is the best possible for each Ω .

Throughout this paper, $\|\cdot\|$ denotes the $L^2(\Omega)$ norm. The gradient ∇ and the Laplacian Δ are understood in the distributional sense. The homogeneous Sobolev space $\hat{H}_0^1(\Omega)$ is defined to be the completion of $C_0^\infty(\Omega)$ in the Dirichlet norm $\|\nabla \cdot\|$.

Inequalities of this type are used in the study of nonlinear partial differential equations (see [4, p. 299], [7, p. 12] and [8]). For bounded domains with smooth boundaries, an inequality of the form of (1), but with a constant depending on the domain, can be obtained by combining the Sobolev inequality

$$(2) \quad \sup_{\Omega} |u| \leq c \|u\|_{H^1(\Omega)}^{1/2} \|u\|_{H^2(\Omega)}^{1/2},$$

with the Poincaré inequality, and the a priori estimate

$$(3) \quad \|u\|_{H^2(\Omega)} \leq c \|\Delta u\|.$$

The inequality (2) has been proven for domains that satisfy a weak cone condition [1]. The estimate (3) has been proven for domains with $C^{1,1}$ boundaries or convex