

The Morse-Sard Theorem with Mixed Differentiability

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ABSTRACT. Suppose that $f : M^m \times \Omega^k \rightarrow N^n$ is of differentiability class $C^{r \times \omega}$ in the sense that $f = f(x, y)$ is jointly C^r in x and analytic in y . If $r \geq m$ it is shown that the Morse-Sard conclusion remains true—the set of critical values of f has n -measure zero. In this context the inequality $r \geq m$ is sharp; the value of k is irrelevant.

In some recent work on geodesic flows, R. Feres and A. Katok (1990) make use of the Morse-Sard Theorem. Their application requires C^∞ differentiability since they deal with maps between manifolds of arbitrarily different dimensions, and the usual hypothesis on a C^r map $f : M^m \rightarrow N^n$ which guarantees that the set of critical values has measure zero is that

$$(M-S) \quad r \geq \max(m - n + 1, 1).$$

However, the maps involved are actually analytic in most of the variables, and here we formulate a version of the Morse-Sard Theorem with minimal smoothness hypotheses in this situation. We believe it has intrinsic interest. The idea that the dimension of the space of variables in which the function is analytic does not enter in the differentiability restriction. We say that a function $f = f(x, y)$ is of class $C^{r \times \omega}$ if it is C^r in x and analytic in y . Locally this means that f is given as a power series in y whose coefficients are C^r functions of x ,

$$f(x, y) = \sum_{\alpha} a_{\alpha}(x)(y - y_0)^{\alpha}.$$

The coefficient $a_{\alpha}(x)$ takes values in the space of symmetric p -linear functionals. We require that the convergence is locally C^r -uniform in the sense that for some $\delta > 0$, the series $\sum \|a_{\alpha}\|_r \delta^{\alpha}$ converges. If the sum is finite, this property is obvious. In general, it can be seen that the radius of convergence depends upper semi-continuously on x and differentiation of f is done termwise respecting both x and y .