

# *The Cyclic Behavior of Translation Operators on Hilbert Spaces of Entire Functions*

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*Dedicated to the memory of Domingo Herrero*

ABSTRACT. We show that the translation operator

$$T : f(z) \rightarrow f(z + 1),$$

acting on certain Hilbert spaces consisting of entire functions of slow growth, is *hypercyclic* in the sense that for some function  $f$  in the space, the orbit  $\{T^n f\}_0^\infty$  is dense. We further show that the operator  $T - I$  can be made compact, with approximation numbers decreasing as quickly as desired, simply by choosing the underlying Hilbert space to be sufficiently small. This shows that hypercyclic operators can arise as perturbations of the identity by “arbitrarily compact” operators. Our work extends that of G. D. Birkhoff (1929), who showed that  $T$  is hypercyclic on the Fréchet space of *all* entire functions, and it complements recent work of Herrero and Wang, who were the first to discover that perturbations of the identity by compacts could be hypercyclic.

**Introduction.** This paper originates from two sources, one old and one new. The old one is G. D. Birkhoff’s intriguing observation about the orbits of translation operators acting on the space of entire functions [Bir, 1929]. To state this result precisely, we introduce the space  $\mathcal{E}$  of entire functions of one complex variable, endowed with the topology of uniform convergence on compact subsets of the plane, and the operator  $T_a : \mathcal{E} \rightarrow \mathcal{E}$  of “translation by the complex number  $a$ ”, defined by:

$$T_a f(z) = f(z + a) \qquad (f \in \mathcal{E}, z \in \mathbf{C}).$$