

Riemannian Foliations on Compact Hyperbolic Manifolds

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Recall that a foliation \mathcal{F} of a Riemannian manifold M is called *metric* or *Riemannian* if its leaves are locally everywhere equidistant. One naturally expects the existence of such a foliation to imply that M possesses a certain amount of symmetry. Interestingly, a number of recent results suggest these are precisely the cases which are rigid, at least if M is compact. For example, Gromoll and Grove have classified all metric foliations of dimension ≤ 3 on Euclidean spheres [6]. As a consequence, all Riemannian fibrations of S^n are congruent to Hopf fibrations if $n \neq 15$. Similarly, any Riemannian submersion from a complete flat manifold onto a compact one is locally the projection of a metric product onto one of the factors [11].

In this note, we use an ergodic theorem to show that any metric foliation of a compact manifold with constant negative curvature $-\kappa$ must be flat, in the sense that the orthogonal (horizontal) distribution is integrable. As a consequence, these manifolds do not admit Riemannian fibrations. We wish to thank the referee for kindly showing us how to improve a previous version of Theorem 3.5.

1. The tangent bundle of the horizontal distribution. Let M be a Riemannian n -manifold. A k -dimensional metric foliation \mathcal{F} on M induces an orthogonal splitting $TM = \mathcal{H} \oplus \mathcal{V}$ of the tangent bundle TM , with \mathcal{V} (the ‘vertical’ distribution) tangent to the leaves, and \mathcal{H} (the ‘horizontal’ distribution) $= \mathcal{V}^\perp$. X, Y will denote horizontal vector fields, T, U vertical ones, and a general vector E decomposes as $E = E^h + E^v \in \mathcal{H} \oplus \mathcal{V}$. Recall that the geometry of \mathcal{F} is locally determined by two tensor fields: the O’Neill tensor is the $(2,1)$ tensor field $A : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{V}$ given by

$$(1.1) \quad A_X Y = (\nabla_X Y)^v = \frac{1}{2} [X, Y]^v,$$

and the second fundamental tensor of the fibers is the tensor field $S : \mathcal{H} \times \mathcal{V} \rightarrow \mathcal{V}$ with

$$(1.2) \quad S_X U = (\nabla_U X)^v.$$

\mathcal{F} is said to be *flat* if $A \equiv 0$ —or equivalently if the horizontal distribution is integrable—and *totally geodesic* if $S \equiv 0$ —or equivalently if the leaves of \mathcal{F} are