

# *Existence and Nonexistence Results for Semilinear Equations on the Heisenberg Group*

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**1. Introduction.** The aim of this paper is to establish existence, regularity and nonexistence results for the problem

$$(1.1) \quad \begin{cases} \Delta_{\mathbb{H}^n} u + f(u) = 0 & \text{in } D \\ u|_{\partial D} = 0 \end{cases}$$

where  $D$  is an open (bounded or unbounded) subset of the Heisenberg group  $\mathbb{H}^n$  and  $\Delta_{\mathbb{H}^n}$  is the subelliptic Laplacian on  $\mathbb{H}^n$ . We recall that  $\mathbb{H}^n$  is the Lie group whose underlying manifold is  $\mathbb{C}^n \times \mathbb{R}$ ,  $n \in \mathbb{N}$ , endowed with the group law

$$(1.2) \quad (z, t) \circ (z', t') = (z + z', t + t' + 2 \operatorname{Im} z \cdot z'),$$

where for  $z, z' \in \mathbb{C}^n$  we have let  $z \cdot z' = \sum_{j=1}^n z_j \overline{z'_j}$ . Setting  $z_j = x_j + iy_j$ , then  $(x_1, \dots, x_n, y_1, \dots, y_n, t)$  form a real coordinate system for  $\mathbb{H}^n$ . In this coordinate system the vector fields

$$(1.3) \quad \begin{aligned} X_j &= \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \\ Y_j &= \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \end{aligned} \quad j = 1, \dots, n,$$

generate the real Lie algebra of left-invariant vector fields on  $\mathbb{H}^n$ . It is easy to check that  $[X_j, Y_k] = -4\delta_{jk} \frac{\partial}{\partial t}$ ,  $j, k = 1, \dots, n$ , and that all other commutators are trivial.

The subelliptic Laplacian is defined as

$$(1.4) \quad \Delta_{\mathbb{H}^n} = \sum_{j=1}^n (X_j^2 + Y_j^2).$$

$\Delta_{\mathbb{H}^n}$  is invariant w.r.t. left-translations. Furthermore, there is a natural group of dilations on  $\mathbb{H}^n$  given by

$$(1.5) \quad \delta_\lambda(z, t) = (\lambda z, \lambda^2 t), \quad \lambda > 0,$$