

On the Structure of the Conformal Scalar Curvature Equation on \mathbf{R}^n

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ABSTRACT. In this paper we wish to conduct a thorough study of the conformal scalar curvature equation ((1.1) below) in \mathbf{R}^n , $n \geq 3$, in case the prescribed scalar curvature function K is negative. First, we establish the existence and uniqueness of the maximal positive solution. Then a complete classification of all possible positive solutions is obtained if K behaves like $-|x|^{-\ell}$ near ∞ for some constant $\ell > 2$.

1. Introduction. In this paper we shall investigate the structure of the set of all positive solutions of the equation

$$(1.1) \quad \Delta u + Ku^p = 0$$

in \mathbf{R}^n , $n \geq 3$, where $\Delta = \sum_{i=1}^n \partial^2 / \partial x_i^2$, $p > 1$, and $K \leq 0$ is a given smooth function on \mathbf{R}^n . Due to the geometric significance of Equation (1.1) (see next paragraph), *we shall restrict ourselves to positive solutions of (1.1) in the entire space \mathbf{R}^n throughout this paper.* The main results here include the existence and uniqueness of a *maximal positive solution* of (1.1), and, in some important cases, a *complete classification* of all positive solutions of (1.1) in \mathbf{R}^n .

Equation (1.1) arises in Riemannian geometry. Let (M, g) be a Riemannian manifold of dimension n and K be a given smooth function on M . The following question has been posed: Can one find a new metric g_1 on M such that K is the scalar curvature of g_1 and g_1 is *conformal* to g ? Since the conformality requirement on g_1 is equivalent to the existence of a positive smooth function φ on M such that $g_1 = \varphi g$, the problem is reduced to finding positive solutions of the elliptic equation

$$(1.2) \quad \frac{4(n-1)}{n-2} \Delta_g u - ku + Ku^{(n+2)/(n-2)} = 0$$

on M , where Δ_g is the Laplace-Beltrami operator on M in the g -metric and k is the scalar curvature of (M, g) , if we write $\varphi = u^{4/(n-2)}$. In the special case