

On Quasiconvex Functions Which Are Homogeneous of Degree 1

STEFAN MÜLLER

ABSTRACT. We show that there exist quasiconvex functions which are homogeneous of degree one but not convex. We also show that for quasiconvex functions f with linear growth $\lim_{t \rightarrow \infty} \frac{f(tA)}{t}$ need not exist for all A .

1. Introduction. A continuous function $f : M^{m \times n} \rightarrow \mathbf{R}$ on the space of (real) m by n matrices is called *quasiconvex* if for some open set $D \subset \mathbf{R}^n$ one has

$$(1) \quad \int_D f(F + \nabla \varphi) \, dx \geq \int_D f(F) \, dx \quad \forall F \in M^{m \times n}, \forall \varphi \in W_0^{1,\infty}(D; \mathbf{R}^m).$$

By a simple covering argument one verifies that if (1) holds for one open set D it holds in fact for all open sets. Clearly convexity implies quasiconvexity but the converse is false if $n \geq 2$ and $m \geq 2$.

The importance of quasiconvexity in the calculus of variations was first realised by Morrey ([Mo 52], [Mo 66]) who showed that quasiconvexity of f is necessary and sufficient for the lower semicontinuity of the functional

$$u \mapsto \int_D f(\nabla u(x)) \, dx$$

with respect to weak-* convergence in $W^{1,\infty}(D; \mathbf{R}^m)$; see Acerbi and Fusco [AF 84] or Marcellini [Ma 85] for further developments.

As (1) is a nonlocal condition it is in general very difficult to verify. This led to the consideration of the following related notions. A function f as above is called *rank-1 convex* if its restriction to any segment whose endpoints differ by a matrix of rank 1 is convex; it is called *polyconvex* if it can be written as a convex function of the minors (subdeterminants) of F . One has the implications (see [Mo 66], [Ba 77], [Da 89]):