

# *A Functional Equation*

RICHARD ROCHBERG & LEE A. RUBEL

*Dedicated to the memory of Ellis R. Kolchin*

ABSTRACT. All entire solutions to the functional equation  $h(x+y)k(x-y) = A(x)B(y) + C(x)D(y)$  are found. The solutions involve (possibly degenerate) Weierstrass sigma functions, or, equivalently, theta functions. The first proof is computational and the second proof is geometrical. Possible extensions with more terms on the right-hand side are briefly discussed. The equation arises in quantum theory, and says that a certain generalized Hankel operator on Fock space has rank not exceeding two.

**1. Introduction.** In this paper, we completely solve the functional equation

$$(1.1) \quad h\left(\frac{x+y}{2}\right)k\left(\frac{x-y}{2}\right) = A(x)B(y) + C(x)D(y),$$

under the assumption that  $h, k, A, B, C, D$  are entire functions of one complex variable. Work on this equation was motivated by a paper [JPW] about operators on the Fock space which had integral kernels of the form  $h\left(\frac{x+y}{2}\right)k\left(\frac{x-y}{2}\right)$ . Such an operator has rank  $\leq n$  if and only if

$$(1.2) \quad h\left(\frac{x+y}{2}\right)k\left(\frac{x-y}{2}\right) = \sum_{i=1}^n A_i(x)B_i(y).$$

Equation (1.1) is the special case of Equation (1.2) in which  $n = 2$ . The case  $n = 1$  of (1.2) is easy, gives only elementary solutions, and is left as an exercise for the reader. The most general equation we are aware of in this direction is

$$(1.3) \quad \sum_{i=1}^m h_i\left(\frac{x+y}{2}\right)k_i\left(\frac{x-y}{2}\right) = \sum_{j=1}^n A_j(x)B_j(y).$$

Attacks on (1.2) in the case  $n = 3$ , by the methods of this paper founder because a key differential equation (see (2.4)), which is reasonable in the case  $n = 2$ , takes five pages of computer printout just to write down in the case  $n = 3$ .