

Spectral Theory of Operators of Hankel Type, I

JAMES S. HOWLAND

ABSTRACT. The spectral theory of the positive self-adjoint operator

$$Hf(x) = \int_0^\infty \frac{\varphi(x)\bar{\varphi}(y)}{x+y} f(y) dy$$

on $L_2(0, \infty)$ and its matrix generalization is obtained when $\varphi(x)$ has limits at zero and infinity. This operator is unitarily equivalent to a Hankel matrix.

Introduction. This paper is devoted to a study of the spectral theory of the non-negative self-adjoint operator

$$H(\varphi)f(x) = \int_0^\infty \frac{\varphi(x)\bar{\varphi}(y)}{x+y} f(y) dy$$

on $L_2(0, \infty)$, and its generalization to matrix-valued $\varphi(x)$. In the case in which the limits $a = \varphi(0+)$ and $b = \varphi(\infty)$ exist, we shall obtain, under mild technical conditions, the following rather complete description:

- (a) $H(\varphi)$ has no singular continuous part;
- (b) its absolutely continuous part is the direct sum of the multiplications by x on $[0, \pi a^2]$ and $[0, \pi b^2]$; and
- (c) the non-zero eigenvalues can accumulate only at zero. If zero is an eigenvalue, it has infinite multiplicity.

The operator $H(\varphi)$ arises formally as the solution of the *Lyapunov equation*

$$AH + HA = (\cdot, \varphi)\varphi,$$

where $Af(x) = xf(x)$. This equation arises in the formalism of “balanced realizations,” which has been used in various inverse problems in the theory of Hankel matrices [10]. An argument from this theory shows that $H(\varphi)$ is unitarily equivalent to the Hankel operator

$$Kf(x) = \int_0^\infty K(x+y)f(y) dy,$$