

# On the Sobolev Quotient of an Einstein Manifold

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Let  $(M^n, g)$  be a closed compact Riemannian manifold of dimension  $n \geq 3$ . For a function  $\varphi$  defined on  $(M, g)$ , we consider the functional

$$(1) \quad Q(\varphi) = \frac{\int_M \left( |\nabla \varphi|^2 + \frac{n-2}{4(n-1)} R_g \varphi^2 \right) dv_g}{\left( \int_M |\varphi|^{2n/(n-2)} dv_g \right)^{(n-2)/n}}$$

where  $R_g$  is the scalar curvature of  $(M, g)$ , and  $dv_g$  represent the Riemannian measure on  $M$  induced by the metric  $g$ .

We define the Sobolev quotient of  $M$  as

$$(2) \quad Q(M) = \inf \{ Q(\varphi) \mid \varphi \in H^1(M), \varphi \neq 0 \text{ on } M \}.$$

A consequence of the work of Aubin [A] and Schoen [S] on the solution of the Yamabe problem for closed compact manifolds is that

$$(3) \quad Q(M) < Q(S^n)$$

when  $(M, g)$  is not conformally equivalent to  $(S^n, g_0)$ , where  $g_0$  is the standard round metric.

It is clear that by perturbing the standard metric of  $S^n$  we can construct manifolds with Sobolev quotient arbitrarily close to the Sobolev quotient of the sphere. However, it is unclear to decide whether or not the Sobolev quotient of an arbitrary compact Einstein manifold can be arbitrarily close to  $Q(S^n)$ . By using a compactness method, we show the following.

**Theorem 1.** *Let  $\mathcal{E}_n$  denote the space of  $n$ -dimensional compact Einstein manifolds, which are not conformally diffeomorphic to  $S^n$ ,  $n \geq 3$ . Then there exists  $\varepsilon(n) > 0$  such that*

$$\sup_{(M, g) \in \mathcal{E}_n} Q(M) \leq Q(S^n) - \varepsilon.$$