

Hausdorff and Box Dimensions of Certain Self-Affine Fractals

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ABSTRACT. The Hausdorff–Besicovich and Bouligand–Minkowski (box) dimensions δ_H and δ_B are computed for a class of self-affine sets. Necessary and sufficient conditions are given for $\delta_H = \delta_B$; it is found that typically $\delta_H \neq \delta_B$. The methods are largely probabilistic, with certain exponential families of probability measures playing a prominent role.

1. Introduction. Let S_1, S_2, \dots, S_r be contractions of \mathbb{R}^2 , i.e., each $S_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ has Lipschitz constant < 1 , the Lipschitz constant being defined as $\text{Lip}(S_i) = \sup\{|S_i x - S_i y|/|x - y| : x \neq y\}$. A result of Hutchinson [Hu] states that there exists a unique nonempty compact subset Λ of \mathbb{R}^2 such that

$$\Lambda = \bigcup_{i=1}^r S_i(\Lambda);$$

we will refer to Λ as the limit set of the semigroup generated by S_1, S_2, \dots, S_r . Many interesting “fractal” sets arise in this manner, and questions concerning the Hausdorff–Besicovich and Bouligand–Minkowski (box) dimensions δ_H and δ_B of Λ are of considerable interest (see [Ma] for definitions). We shall discuss a number of such questions in the case where the contractions S_i are affine mappings of a certain special type. (Note: For similarities S_i , $\delta_H = \delta_B$ and δ_H is the “similarity dimension”: see [Hu].)

Our interest stems from papers of Bedford [Be], McMullen [Mc], and Falconer [Fa]. McMullen studied the special case in which each S_i has the form

$$S_i(x) = \begin{pmatrix} n^{-1} & 0 \\ 0 & m^{-1} \end{pmatrix} x + \begin{pmatrix} k_i/n \\ \ell_i/m \end{pmatrix},$$

where $1 < m < n$, $0 \leq k_i < n$, and $0 \leq \ell_i < m$ (k_i and ℓ_i are integers); thus each S_i maps the unit square onto an $n^{-1} \times m^{-1}$ rectangle R_i contained in the unit square. He found formulas for δ_H and δ_B and discovered that $\delta_H = \delta_B$ only in exceptional circumstances, namely, when all “rows” that contain at least one R_i , contain the *same* number of R_i (i.e., if t_ℓ is the number of i with $\ell_i = \ell$, for