

The Corona Theorem and Interpolating Blaschke Products

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ABSTRACT. We prove the following:

An interpolating Blaschke product $B(z)$ on the unit disc D is in the ideal generated by $f_1(z), f_2(z), \dots, f_n(z)$ in $H^\infty(D)$ if and only if $|B(z)| \leq C(|f_1(z)| + |f_2(z)| + \dots + |f_n(z)|)$ for all $z \in D$.

0. Introduction. Let D be the unit disc in the complex plane \mathbb{C} and $H^\infty = H^\infty(D)$ the ring of bounded analytic functions on D . In 1962, Carleson [4] proved the following famous corona theorem:

Let f_1, f_2, \dots, f_n be functions in H^∞ such that

$$|f_1(z)| + |f_2(z)| + \dots + |f_n(z)| \geq \delta > 0 \quad \text{for all } z \in D.$$

Then there are H^∞ functions g_1, g_2, \dots, g_n so that

$$f_1(z)g_1(z) + f_2(z)g_2(z) + \dots + f_n(z)g_n(z) = 1.$$

One can view the corona theorem as a theorem on the condition when the function 1 is in the ideal generated by f_1, f_2, \dots, f_n . Rubel (see [9]) thus raised the following problem: Given g, f_1, f_2 in H^∞ , when is g in $I(f_1, f_2)$, the ideal generated by f_1 and f_2 ? An obvious necessary condition is that $|g(z)| \leq C(|f_1(z)| + |f_2(z)|)$ for all $z \in D$. However, an example due to Rao [8] showed that this is not sufficient. It runs as follows. Take two Blaschke products $B_1(z)$ and $B_2(z)$ whose zeros are distinct and are so close that $\inf_{z \in D} (|B_1(z)| + |B_2(z)|) = 0$, and let $g(z) = B_1(z)B_2(z)$ and $f_1(z) = B_1^2(z)$, $f_2(z) = B_2^2(z)$. Then $|g(z)| \leq |f_1(z)| + |f_2(z)|$. Suppose there were H^∞ functions g_1 and g_2 such that $g(z) = f_1(z)g_1(z) + f_2(z)g_2(z)$, then $B_1(z)B_2(z) = B_1^2(z)g_1(z) + B_2^2(z)g_2(z)$, hence $g_1 = B_2h_1$, $g_2 = B_1h_2$ for some $h_1, h_2 \in H^\infty$. Therefore, $1 = B_1(z)h_1(z) + B_2(z)h_2(z)$, but this contradicts the corona theorem. A natural question arises: what if the zeros of $g(z) = B_1(z)B_2(z)$ are not so close, say for example, g is an interpolating Blaschke product? Our theorem provides the positive answer.