

# *Equivalence of Homogeneous Principal Hardy Submodules*

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Recently, there has been an approach to operator theory using the language of Hilbert modules [8]. In this sense, the Hardy space  $H^2(\mathbf{D}^n)$  of the polydisk  $\mathbf{D}^n$  is a Hilbert module over the polydisk algebra  $A(\mathbf{D}^n)$ , where every function in  $A(\mathbf{D}^n)$  acts on  $H^2(\mathbf{D}^n)$  by multiplication. Submodules of  $H(\mathbf{D}^n)$  are exactly invariant subspaces. Two submodules  $H_1, H_2$  are called unitarily equivalent (similar, quasi-similar) if there exists a unitary (invertible, quasi-affine) module maps between  $H_1$  and  $H_2$ . In this paper we will examine the conditions under which two submodules, each of which is generated by a single homogeneous polynomial, are unitarily equivalent (similar, quasi-similar). The main results are Theorem 1 and Theorem 2. When  $n = 1$ , the Beurling Theorem shows that every two submodules of  $H^2(\mathbf{D}^n)$  are unitarily equivalent. Therefore, the above question is higher dimensional in nature.

In our earlier joint work [9], we showed that quite the opposite of the Beurling Theorem was true. Namely, if two submodules  $H_1, H_2$  have the property that the sets of common zeros are of co-dimension at least two, and if  $H_1, H_2$  have the same boundary singular distribution, then quasi-similarity of  $H_1, H_2$  forces  $H_1 = H_2$ . As a consequence, for any two ideals  $I_1, I_2$  of the polynomial ring  $\mathbf{C}[z]$  with height at least 2, if  $I_1$  and  $I_2$  have different zero sets in  $\mathbf{D}^n$ , then their closures  $[I_1], [I_2]$  in  $H^2(\mathbf{D}^n)$  cannot be quasi-similar. A more algebraic approach to this problem when  $\mathbf{D}^n$  is replaced by arbitrary domains in  $\mathbf{C}^n$  is contained in [19]. We also refer the interested readers to [1], [2] for more developments on this topic. Most of the above mentioned results require that the co-dimension of the zero set of the submodule in question be at least 2. From an algebraic point of view this implies that the submodules are not singly generated. In this paper we will focus on those submodules generated by a single homogeneous polynomial. We give necessary and sufficient conditions for two such submodules to be similar, quasi-similar and unitary equivalent. As a corollary, we find a large class of singly generated submodules which are mutually not quasi-similar. This shows, again, that the Beurling Theorem is not valid in higher dimension, even when the submodules in question are singly generated.

This paper is organized as follows. In Section 1, we list some function-theoretic preparations. The main reference is [15]. Similarity and quasi-similarity