

# Nearly Uniform Convexity of Infinite Direct Sums

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ABSTRACT. Those kinds of infinite direct sums are characterized for which NUC (NUS, respectively) is inherited from the component spaces to the direct sum.

**1. NUC and the lower KK-modulus.** The Banach space  $X$  is said to be *nearly uniformly convex*, abbreviated NUC (cf. [2]), if: for every  $\varepsilon > 0$ , there exists  $\delta > 0$  such that the convex hull  $\text{conv}\{x_n\}$  of every sequence  $\{x_n\}$  in the unit ball  $B_X$  of  $X$  with *separation*

$$\text{sep}\{x_n\} = \inf_{n \neq m} \|x_n - x_m\|$$

not smaller than  $\varepsilon$  contains an  $x$  with  $\|x\| \leq 1 - \delta$ .

It is well known (cf. [2]) that  $X$  is NUC iff  $X$  is reflexive and UKK, where  $X$  is said to be *uniformly Kadec-Klee*, abbreviated UKK, if: for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $\|x\| \leq 1 - \delta$  whenever  $x$  is the weak limit of a sequence in  $B_X$  with separation not smaller than  $\varepsilon$ .

This characterization motivates the following definition of the function  $\Delta_X : [0, 2] \rightarrow [0, 1]$  that we call the *KK-modulus* of  $X$ :

$$\Delta_X(\varepsilon) = \inf\{1 - \|x\| \mid x_n \rightharpoonup x, \{x_n\} \subset B_X, \text{sep}\{x_n\} \geq \varepsilon\}.$$

Here, we agree to define  $\Delta_X(\varepsilon) = 1$  if the right hand side infimum ranges over the empty set (i.e., we take the infimum in the ordered set  $[0, 1]$ ). This means, that in case of a finite dimensional space  $X$ , we have

$$\Delta_X(\varepsilon) = \begin{cases} 0, & \varepsilon = 0, \\ 1, & 0 < \varepsilon \leq 2. \end{cases}$$

In the definition of the KK-modulus and also in the sequel, the symbol  $\rightharpoonup$  denotes convergence in the weak-topology.