

On a Result of Beurling

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Dedicated to Nobuyuki Suita on the occasion of his sixtieth birthday

1. The well-known theorems of Fatou and F. and M. Riesz give (in their primary form) for non-constant bounded regular functions in the unit disc the characterization of those sets on the boundary of the disc where respectively radial limits can fail to exist or radial limits can have a given value, the operative condition being to have measure zero. Beurling [1] considered the same problems for non-constant meromorphic functions mapping the unit disc on a Riemann surface of finite spherical area. He showed that sets where the radial limit failed to exist would have capacity zero. However, he was able to prove the corresponding condition in the second problem only when the Riemann surface had an appropriate behavior in the neighborhood of the limit value [1, p. 12] (see also [4, p. 121]). The fact is that, in the presence of such a local condition, if the problem is properly formulated, the global behavior of the Riemann surface is completely irrelevant. Some indication of this appears already in the work of Tsuji [4], [5; p. 349] who showed that one could drop the condition of finite spherical area but only with the addition of the condition that the function take some three values at most finitely often. This is of course a normality condition. Moreover, Beurling's local condition, used also by Tsuji, implies that the surface has locally finite area. It will be seen that this condition is unnecessarily restrictive.

2. Let $f(z)$ be meromorphic in the unit disc D . We say a value α on the sphere satisfies Condition C for f at the point P on the unit circumference ∂D if α is a cluster value of f along every path tending to P in D .

Theorem. *Let $f(z)$ be a non-constant meromorphic function in the unit disc D mapping D onto the Riemann surface \mathcal{S} . For the value α let ζ be a local uniformizing parameter on the sphere representing α as the origin and let $h(r)$ be a continuous non-negative real-valued function defined for $0 < r \leq T$ for which there exists a measurable subset τ of $(0, T)$ such that $\int_{\tau} h(r) dr = \infty$. Let $\sigma(\lambda)$, $0 < \lambda < T$ denote the subset of \mathcal{S} covering the set of points ζ with*

$$|\zeta| = t, t \in \tau \cap (\lambda, T) = \tau(\lambda).$$