

Collapsing to Almost Riemannian Spaces

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ABSTRACT. We establish a convergence result which extends Yamaguchi's Fibration Theorem to almost Riemannian spaces and Plaut's compactness theorem to the collapsing situation. We then apply the convergence result to obtain pinching and structure theorems for Riemannian manifolds.

1. Introduction. The recognition program in Riemannian geometry strives to identify an unknown Riemannian manifold via measurements of its metric invariants (cf. [GM]). The best possible outcome is determination of the isometry class of the manifold (see, for example, [B1] or [GM]). In the absence of that its (differential) topological identification (see, for example, [B2], [GS], or [SY]). When even topological identification is not possible, we might still hope to describe the structure of the manifold (see, for example, [CG1] or [CG2]) or at least to prove that such a manifold can have only finitely many (differential) topological types (see, for example, [AC] or [GPW]).

Since its introduction in [G1], the Gromov-Hausdorff convergence technique has had many applications in the recognition program in bounded geometry (see, for example, [B2] or [D]), and recently it has had applications in the study of manifolds with only a lower curvature bound (see, for example, [GP1], [GPW], or [Y1]). This experience has shown that the following problem is of fundamental importance.

Convergence Problem. *Given $k \in \mathbb{R}$ and $n \in \mathbb{N}$ let $\mathcal{M}_k(n)$ denote the class of closed, connected, Riemannian n -manifolds with sectional curvature $\geq k$. If a sequence of spaces $\{Y_i\}_{i=1}^\infty$ in the Gromov-Hausdorff closure of $\mathcal{M}_k(n)$ (denoted $GHC\mathcal{M}_k(n)$) converge to a space X , then, if i is large, how are the geometries and topologies of the Y_i 's related to the geometry and topology of X ?*

We restrict our attention to the class $\mathcal{M}_k(n)$ because in complete generality, the hypothesis $\lim_{G-H} Y_i = X$ does not imply any relationship between the topologies of the Y_i 's and the topology of X . Indeed, any compact metric space is the limit of a sequence of discrete spaces, and any compact length space is the limit of surfaces isometrically embedded in \mathbb{R}^3 (see [GLP] and [C] respectively). Even if the Y_i 's and X are connected, Riemannian manifolds of the same dimension, without further restrictions on the spaces there can be no additional comparison between the topologies of the Y_i 's and the topology of X .