

Three Results Concerning the Support of Functions and Their Fourier Transforms

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ABSTRACT. In this paper we offer a method for constructing functions with many gaps in their supports and in the supports of their Fourier transform. As a result we answer a question of Benedicks [Ben1] and we present the construction of a characteristic function with Fourier transform vanishing on an interval. We also give a new proof of a theorem of Benedicks which says what systems of intervals may support a function with Fourier transform vanishing on an interval. We slightly strengthen Benedicks result [Ben1].

1. INTRODUCTION

Various aspects of the interaction between the smallness of the support of a given function f on \mathbb{R} and the smallness of the support of its Fourier transform have been treated by many authors.

What we mean by support will vary in what follows. But only two cases will occur, namely: the closed support or just $\{t : f(t) \neq 0\}$.

Following [Ben1] here is the most general statement of the problem: *Suppose that f is a function or tempered distribution supported on a set A in \mathbb{R}^d or \mathbb{T}^d or \mathbb{Z}^d such that its Fourier transform \hat{f} is supported on B (in \mathbb{R}^d , \mathbb{Z}^d , or \mathbb{T}^d). Then given some a priori conditions on f and \hat{f} , which conditions on A , B imply that $f \equiv 0$, and conversely, when does there exist a non-zero function or distribution such that $\text{supp } f \subset A$, $\text{supp } \hat{f} \subset B$?*

In this form the problem is certainly too general. As a particular case it includes the description of zero-sets for classes of analytic functions in the unit disc \mathbb{D} with prescribed smoothness up to the boundary \mathbb{T} . This latter problem has been considered during the last 40 years, starting from the paper [C] of Carleson.

We mention that the case $A = \mathbb{T} \setminus I$, where I is an interval and $a = |A| < 2\pi$, is treated in [Lev] in detail. By duality, such uniqueness is equivalent to