

A Harnack Inequality Approach to the Interior Regularity of Parabolic Equations

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We continue our investigations on interior regularities. Here we extend the results of [CW] to flows of curvatures.

Introduction. In this paper, we prove $C^{1,\alpha}$ regularity for evolution $S(t)$ of surfaces whose normal velocity S_t equals to a function of its curvature tensor \mathbf{II} , normal vector ν , position vector X and time t :

$$(1) \quad S_t - F(\mathbf{II}, \nu, X, t) = R(\nu, X, t),$$

where the curvature tensor \mathbf{II} is the second fundamental form in the space direction only and ν its normal in the space direction.

We assume that F is uniformly elliptic in \mathbf{II} , Lipschitz in ν and small oscillation in (X, t) in the space of L^{n+1} and that R is locally bounded in ν and decays in L^{n+1} in the variable (X, t) . Under these conditions, we show that every Lipschitz solution S of (1) is locally $C^{1,\alpha}$.

By the compactness arguments in [C] and [W2], we see that the $C^{1,\alpha}$ regularity for (1) actually reduces to $C^{1,\alpha}$ regularity for much simpler equations of the type

$$(2) \quad S_t - F(\mathbf{II}, \nu) = 0.$$

See Theorem 1.3 in [W2] for details.

Main Theorem. *Suppose F satisfies uniformly ellipticity \mathbf{II} and Lipschitz in ν as in [CW]. Assume S is, in the sense of (3), a Lipschitz solution of (2) in the cylinder $C_1 = Q_1(0, 0) \times R^1$. Then $S \in C^{1,\alpha}(C_{1/2}(0, 0))$. Moreover $\|S\|_{C^{1,\alpha}(C_{1/2})}$ has an upper bound, depending only on $\|S\|_{\text{Lip}(C_1)}$ and $\|S\|_{C^{1,\alpha}(C_{1/2})} \leq C[S]_{\text{Lip}}$ for $[S]_{\text{Lip}} \leq 1$ and some $\alpha > 0$.*

In general, we write $C_r(x, t) = Q_r(x, t) \times R^1$.

We refer the reader to [CW] for a more heuristic discussion and reference therein.