

# *Stability of Flows of Ideal Fluids and the Geometry of the Group of Diffeomorphisms*

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ABSTRACT. We are concerned with the stability of flows of an ideal fluid in Lagrangian coordinates. We find conditions for linear and nonlinear stability of flows in the Lagrangian sense and construct classes of flows stable in the Eulerian but unstable in the Lagrangian sense as well as flows unstable in both senses. Our method involves the study of the geometry of  $\mathcal{D}_\mu(M)$ —the group of volume preserving diffeomorphisms of a smooth, compact,  $n$ -dimensional Riemannian manifold  $M$  (in particular a bounded domain in  $R^n$ ), which is the region filled with fluid. Related to stability is the existence of conjugate points on  $\mathcal{D}_\mu(M)$ . Using Jacobi fields we construct such points for spheres  $S^2$ ,  $S^3$  and the ball  $B^3$ .

## 1. INTRODUCTION

In the Lagrangian formulation of hydrodynamics of ideal fluids, as developed by V. Arnold [A], D. Ebin and J. Marsden [EM],  $\mathcal{D}_\mu(M)$ —the group of volume preserving diffeomorphisms of a compact Riemannian manifold  $M$ —can be considered an infinite dimensional submanifold of the weak Riemannian manifold  $\mathcal{D}(M)$ , the group of all diffeomorphisms of  $M$ . It is well known that fluid flows in  $M$  (with or without boundary) correspond to geodesics in  $\mathcal{D}_\mu(M)$ . Using the weak Riemannian structure we introduce the notion of Lagrangian stability of fluid flows in  $M$  saying that a flow  $\eta(t)$  is stable if all geodesics in  $\mathcal{D}_\mu(M)$  with sufficiently close initial conditions at  $t = 0$  remain close for all  $t \geq 0$ .

Our method is to study the curvature of  $\mathcal{D}_\mu(M)$ . This is because the curvature tensor enters the Jacobi equation which, being the linearized version of the geodesic equation, controls the infinitesimal stability behaviour of geodesics. This approach to the stability of fluids was first suggested in [A], where the complete computations for the curvature tensor of  $\mathcal{D}_\mu(T^2)$ , ( $T^2$  the flat two torus) were also presented. The group theoretical approach of [A] (see also [Lu]) leads,