

Integral Representation of Relaxed Functionals on $BV(\mathbf{R}^n, \mathbf{R}^k)$ and Polyhedral Approximation

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ABSTRACT. We consider the functional

$$F(u, \Omega) = \begin{cases} \int_{\Omega} f(x, u, Du) dx & \text{if } u \in X = W_{\text{loc}}^{1,1}(\Omega, \mathbf{R}^k) \cap C^1(\Omega, \mathbf{R}^k) \\ +\infty & \text{if } u \in L_{\text{loc}}^1(\Omega, \mathbf{R}^k) \setminus X \end{cases}$$

where $\Omega \subset \mathbf{R}^n$ is an open set, $u : \Omega \rightarrow \mathbf{R}^k$, $f(x, z, p)$ is a continuous function, convex with respect to p and with linear growth. We show that the relaxed functional of F , defined by

$$\bar{F}(u, \Omega) = \inf \left\{ \liminf_{h \rightarrow +\infty} F(u_h, \Omega), u_h \in X, u_h \rightarrow u \text{ in } L_{\text{loc}}^1 \right\}$$

has an integral representation on $BV(\Omega, \mathbf{R}^k)$. We give a formula for the energy density, and if the integrand f is *flat* (see Def. 3.4), we compare the density with other integrands already proposed in the literature.

Introduction. In variational problems, when looking for the minima of functionals which fail to be lower semicontinuous, it is interesting to study the behaviour of minimizing sequences; to this end, given such a functional F , a by now standard technique is to introduce the (sequentially) lower semicontinuous envelope (or relaxed functional) defined by

$$\bar{F}(u) = \inf \left\{ \liminf_{h \rightarrow +\infty} F(u_h) \right\},$$

where the infimum is taken over all the sequences (u_h) converging to u in the given topology. Such a technique goes back to H. Lebesgue, and has been subsequently embodied in the theory of Γ -convergence (see [AH], [DGF], [DG1], [DG2] and the wide bibliographies in these papers).