

Direction of Vorticity and the Problem of Global Regularity for The Navier-Stokes Equations

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*Dedicated to our friend and teacher, Ciprian Foias
on the occasion of his sixtieth birthday*

1. Introduction. The problem of global regularity for the three dimensional incompressible Navier-Stokes equations is the following: given smooth, localized, divergence-free initial data, do smooth solutions of the Cauchy problem exist for all time? The answer to this problem is not known. What is known is the existence of global weak solutions ([1]) with certain properties ([2], [3], [4], [5]). If the spatial dimension is two, the problem of regularity has an affirmative answer. The main difference between the two dimensional and three dimensional problems can best be understood by considering the dynamics of the vorticity (curl of velocity). This vector is perpendicular to the plane of motion in the two dimensional case and its magnitude is uniformly bounded. In the three dimensional case, there exists a stretching mechanism for the vorticity magnitude which is nonlinear, and hence potentially capable of producing finite time singularities. In this paper we prove that if the direction of vorticity is sufficiently well behaved in regions of high vorticity magnitude, then the solution is smooth.

Denote by $\varphi(x, y, t)$ the angle between the unit vectors of direction of vorticity at locations x and y at time t . Our assumption is that there exist positive constants Ω and ρ such that, if the magnitude of vorticity at both locations exceeds Ω , then

$$|\sin \varphi(x, y, t)| \leq \frac{|x - y|}{\rho}.$$

This inequality holds in some cases in which the direction of vorticity is not Lipschitz continuous in regions of high vorticity (for instance, if two antiparallel vortex tubes osculate). The process of creation and pairing of antiparallel vortex tubes has been observed in physical and numerical experiments.

There exist viscous mechanisms which produce some degree of regularity of the direction of vorticity ([5], [6], [7]) (see (21)). The gap between this known