

Hilbert–Samuel Polynomials for Hilbert Modules

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During the past decade a systematic study of modules in the context of operator theory was undertaken [4]. A natural function algebra of holomorphic functions operating continuously on a module with a Hilbert space structure captures many of the features of single operator theory while providing a framework for investigating the multivariate case. Early results [5] have demonstrated the value of an eclectic approach combining ideas and techniques from commutative algebra, algebraic geometry and complex geometry with those from functional analysis and operator theory.

Much of the effort so far has been directed toward the study of concrete examples, seeking to classify them. This is particularly important for the multivariate case where the absence of amenable examples has hampered development.

Classification requires the identification of invariants to distinguish between various examples. One effective approach has been based on spectral localization [3], a notion borrowed from commutative algebra. However, more than the approach is borrowed since the broadest rigidity theorem rests on results of Grothendieck and Zariski [10] extended to the Hilbert module context. The synergistic interplay of the techniques in this work is particularly appealing and suggests a deeper connection between the fields than might at first be expected. Understanding better the true situation is one of the goals of the present paper. But of equal importance is the search for effective, computable invariants for multivariate Hilbert modules.

In commutative algebra it is shown in the presence of certain finiteness hypotheses that a canonical polynomial can be defined for a module, thus providing an invariant for the module. However, the story does not stop here. One seeks to interpret the coefficients and degree of the polynomial in terms of algebraic and geometric properties of the module and algebra. In this note we show that a Hilbert–Samuel polynomial can be defined for Hilbert modules if again we impose appropriate finiteness conditions on the function algebra and module. We then study basic properties of this notion and introduce some examples. Our principal result shows in the torsion-free case that we can identify an algebra and a corresponding module for which the usual Hilbert–Samuel polynomial agrees