

The Existence of Eigenvalues of Integral Operators

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Dedicated to Ciprian Foias, with friendship and admiration

1. Introduction. The theory of eigenvalues for compact selfadjoint or normal operators in Hilbert space is well-nigh perfect: the eigenvectors form a complete set. Similar results hold for perturbations of such operators which are small in some sense. Not much is known beyond this, except for Perron-type results, see [1], [3], that positive operators have a positive eigenvalue with a positive eigenfunction. In this note we show how to use Lidskii's celebrated trace formula, see [1] and [3], to ascertain that a given integral operator has a nonzero eigenvalue or not.

2. We assume that the reader is familiar with the theory of trace class operators in Hilbert space. The trace of such an operator K is defined as its matrix trace:

$$(1) \qquad \operatorname{tr} K = \sum_n (K e_n, e_n) ,$$

where $\{e_n\}$ is any orthonormal basis; the value of the trace is independent of the orthonormal basis chosen.

Suppose K is an integral operator, say on the interval $[0, 1]$:

$$(2) \qquad (Kf)(x) = \int k(x, y) f(y) dy ,$$

acting on functions f in $L^2[0, 1]$. If the kernel $k(x, y)$ is smooth enough, the integral operator K given by (2) is of trace class. Furthermore its trace is given by the integral of its diagonal:

$$(3) \qquad \operatorname{tr} K = \int k(x, x) dx .$$