

# *Positive Polynomials on Compact Semi-algebraic Sets*

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*Dedicated to Professor Ciprian Foias on his sixtieth anniversary*

**Introduction.** An elementary argument shows that a non-negative polynomial on the real line can be represented as a sum of squares of polynomials. To prove that a non-negative polynomial depending on two variables is not representable in general as a sum of squares of polynomials is a more delicate question. It was Hilbert who found a non-constructive proof of this fact in 1888 (an example of such a non-representable polynomial was given only in 1967 by Motzkin). Hilbert subsequently reformulated the problem by asking whether a non-negative polynomial can be written as a sum of squares of rational functions, and he succeeded to prove this statement in the case of two variables. As an answer to Hilbert's 17th problem, E. Artin proved, in 1927, that any non-negative polynomial in  $n$  variables is a sum of squares of rational functions. The theory of extensions of closed real fields, which Artin developed for that very purpose, is now one of the fundamental chapters of real algebraic geometry. (See [5] for an introduction to this field and for excellent historical remarks).

The problem of decomposing polynomials which are non-negative on certain subsets of  $\mathbb{R}^n$  into prescribed sums that involve squares of rational functions is still of interest today for a variety of reasons (see for instance [12]). In the present paper we will focus on a very specific question, namely, on decomposing into simpler elements the polynomials which are positive on compact semi-algebraic subsets of the euclidian space. The main method in obtaining these decompositions involves, besides real algebraic arguments, some elementary results of duality theory on locally convex spaces as well as the theory of subnormal operators. It is now standard to treat the decomposition of positive polynomials into a sum of prescribed functions as a question of separating two convex cones by linear functionals. In its turn, this idea brings into discussion the structure of the linear functionals which are represented by positive measures, and hence moment problems. On the other hand, some normal or subnormal operators attached to these measures appear naturally into discussion. Thus, we will rely