

Quasilines and the Hayman-Wu Theorem

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1. Introduction and results . Let $G \subset \mathbf{C}$ be a simply connected domain, \mathbf{D} the open unit disk, $f : G \rightarrow \mathbf{D}$ a conformal map and L a line or a circle. The well-known theorem of Hayman-Wu [HW] says the following:

Theorem HW. *There is a universal constant C so that*

$$(1) \quad \text{length} (f(L)) = \sum_i \text{length} (f(L_i)) \leq C$$

for any choice of G , f and L , where the L_i 's are the components of $L \cap G$.

Proofs simpler than the original one have appeared in [GGJ], [FHM] and [Øy]. Øyma's proof in [Øy] is very short and gives $C \leq 4\pi$, the best bound known so far.

Bishop and Jones, in [BJ1] (see also [BJ2]) have characterized the Jordan curves L for which inequality (1) holds for any choice of G and f , with a constant C depending only on L . These turn out to be the so-called Ahlfors regular curves, i.e., curves L for which there is a constant C so that $\text{length}\{z \in L : |z - a| < r\} \leq Cr$ for every $a \in L$ and r , $0 < r < \text{diam } L$.

To understand the geometry of the Hayman-Wu Theorem, one is naturally lead to ask what is the role of conformality in it. More precisely, under the assumption that f is only K -quasiconformal, one expects that the following analogue of Theorem HW holds:

$$(2) \quad \sum_i \text{diam} (f(L_i)) \leq C(K).$$

This appears as a conjecture in [FHM]. In that direction, Väisälä has shown in [V] the following:

Theorem V. *For every $\varepsilon > 0$, $K \geq 1$ there is a constant $c = c(\varepsilon, K)$, so that for every K -quasiconformal mapping f from a simply connected domain G onto \mathbf{D} and for every line L*

$$(3) \quad \sum_i (\text{diam} (f(L_i)))^{1+\varepsilon} \leq c(K, \varepsilon)$$

where L_i are the components of $L \cap G$.